Stochastic flows on noncompact manifolds

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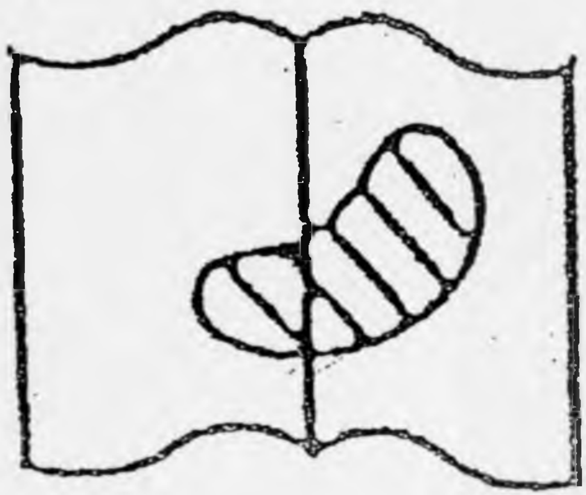

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Abstract

Here we look at the existence of solution flows of stochastic differential equations on noncompact manifolds and the properties of the solutions in terms of the geometry and topology of the underlying manifold itself. We obtain some results on "strong p-completeness" given conditions on the derivative flow, and thus given suitable conditions on the coefficients of the stochastic differential equations. In particular a smooth flow of Brownian motion exists on submanifolds of R^n whose second fundamental forms are bounded. Another class of results we obtain is on homotopy vanishing given strong moment stability. We also have results on obstructions to moment stability by cohomology. Also we obtain formulae for $d(e^{\frac{1}{2}t\Delta^h}\phi)$ for differential form ϕ in terms of a martingale and the form itself, not just its derivative, extending Bismut's formula.

Contents

Part I

Background

Chapter 1

Introduction and preliminaries

1.1 Introduction

Let $\{\Omega, \mathcal{F}, \mathcal{F}_t, P\}$ be a filtered probability space satisfying the usual conditions including right continuity. Let M be a n dimensional smooth manifold. Consider the following stochastic differential equation(s.d.e.) on M:

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt. \tag{1.1}$$

Here B_t is a R^m valued Brownian motion($\mathcal{F}_t$-adapted), X is C^2 from $R^m \times M$ to the tangent bundle TM with $X(x)$: $R^m \to T_xM$ a linear map for each x in M, and A is a C^1 vector field on M.

By $\circ$ we mean the Stratonovich integral.

Let u be a random variable independent of $\mathcal{F}_0$. Denote by $F_t(u)$ the *solution* starting from u, with $\xi(u)$ the *explosion time*. By a solution we mean a maximal solution which is sample continuous unless otherwise stated. Under our assumptions on coefficients the solution to equation 1.1 is unique in the sense of that if (x_t, ξ) and (y_t, η) are two solutions with same initial point, then they are versions of one another, i.e. $\xi = \eta$ almostly surely, and

$x_t = y_t$ almostly surely for all t on $t < \xi$.

Furthermore the solutions to the stochastic differential equation (1.1) are *diffusion* processes, i.e. $\{F_t(x), t \geq 0\}$ is a path continuous strong Markov process for each x. In fact most of the interesting diffusions can be given in this way. The importance of a diffusion process is largely related to its semigroup P_t (given by $P_t f(x) = Ef(F_t(x))\chi_{t<\xi(x)}$), and the corresponding infinitesimal generator $\mathcal{A}$ (which we discuss in detail in chapter 4), or to the consideration of (1.1) as a random perturbation of the dynamical system given by the vector field A.

The pair $((X, A), (B_t, t))$, often simplified as (X, A), is called a *stochastic dynamical system* (s.d.s.).

The thesis contains two themes. The first considers fundamental problems of stochastic differential equations: completeness and strong completeness. The main theme of the second part is to relate geometrical and topological properties of manifolds to the diffusion processes on it. The first furnishes a start for the second. And both make good use of the fact that integrability conditions on the derivative flows give very strong conditions on the manifolds as well as on the diffusion themselves. We have nonexplosion, strong p-completeness results, homotopy and cohomology vanishing results assuming this type of condition.

As we shall see properties of diffusions are directly related to those of the manifolds. An immediate demonstration of the connection between diffusion processes and geometry is given by the well known fact: the nonexplosion of a Brownian motion (which exists naturally on a Riemannian manifold) is equivalent to the uniqueness of solutions of the heat equations, which relates to the behaviour at infinity of the associated probability semigroup as shown in chapter 4.

Before going into detail, we would also like to point out that the existence of a continuous flow is of basic importance in the study of ergodic properties.

The first three chapters of the thesis are preliminary. Of these chapter 1 is to establish notation and contains quoted results, chapter 2 and chapter 3 deal with results which are essentially known (especially in special cases) but not found anywhere in suitable form. Much of the treatment in chapter 2, in which we discuss carefully the Bismut-Witten Laplacian $\triangle^h$ and its associated semigroups, was arrived at together with D. Elworthy and S. Rosenberg. In chapter 3 we look at the invariant measure for an h-Brownian motion and its ergodic properties.

Inspired by the idea of uniform covers, we introduce the weak uniform cover method in chapter 4, which allows us to look at many different problems from the same point of view including the explosion problem, the behaviour at infinity of diffusion processes and probability semigroups, along with the geometry of the underlying manifold at infinity. As an example we conclude that linear growth gives the C_0 property and have a result on the nonexplosion of s.d.e. on an open set of R^n. The results on nonexplosion are used in chapter 5 to obtain strong completeness.

In chapter 5, the concept of strong p-completeness is introduced due to the complexity of strong completeness. A stochastic differential equation is said to be strongly p-complete if it has a version which is smooth on any given smooth singular p-simplex. It is called strongly complete if there is a smooth flow of the solution on the whole manifold. We start with a theorem on strong 1-completeness which is applied to get $dP_tf = \delta P_t(df)$ in chapter 6. Here $\delta P_t(df) = Edf(TF_t\chi_{t<\xi})$, and TF_t is the derivative flow(c.f. section 1.2). Strong 1-completeness is needed in chapter 7 to get homotopy vanishing results and is also used in chapter 8. Later we give a theorem on strong p-completeness, in particular the existence of a continuous flow and flows of diffeomorphisms. Strong p-completeness leads naturally to cohomology vanishing given strong moment stability (see page 118). These theorems are originally given in terms of integrability conditions on TF_t, but those conditions can also be checked in terms of the bounds on the coefficients of the stochastic differential equations. See section 5.3. In particular, as a simple example, there is a smooth flow of Brownian motions on a submanifold of R^n whose second fundamental form is bounded.

In chapter 6, we look at the probability semigroup for 1-forms, the heat semigroup for 1-forms, and the differentiation of the probability semigroups for functions and ask a basic and yet important question: does $dP_t f$ equals $\delta P_t(df)$? This question is answered from different approaches, but all require conditions on TF_t. We use this knowledge of the semigroups extensively in the next two chapters. Following from the discussions, we get a result on the L^p boundedness and contractivity of heat semigroups for forms. As a corollary, we give a cohomology vanishing result.

Proceeding to chapter 7, the interplay between diffusion processes and geometric and topology properties of the underlying manifold becomes clearer. The main theorems here are on the vanishing of the first homotopy group $\pi_1(M)$ given conditions on the regularity of the diffusions and strong moment stability, i.e. the supremum over a compact set of the p^{th} norm of the derivative flow decays exponentially fast. In particular we conclude that a certain class of manifold cannot have a strongly moment stable s.d.s. if its fundamental group is not trivial. We also look briefly at the vanishing of harmonic 1-forms and cohomology vanishing.

Finally in chapter 8, we get a formula for $d(P_t f)$ for elliptic systems in terms of $f(x_t)$ and a martingale, extending the formula given in [26]. We also obtain a similar type of formula for q forms. In particular we have an explicit formula for the gradient of the logarithm of the heat kernel extending Bismut's formula.

Bibliographical notes are scattered in every chapter. But we would like to refer to [31] and [27] for general references. See also [45].

Flow chart

$$\text{various criteria} \xrightarrow{\text{chapter 4}} E\sup_{s\le t}|TF_s| < \infty + \text{complete}$$

$$E\sup_{s\le t}|TF_s| < \infty + \text{complete} \xrightarrow{\text{chap 5}} \text{strong 1-complete}$$

$$E\sup_{s\le t}|TF_s| < \infty + \text{complete} \xrightarrow{\text{chap 6}} dP_t f = \delta P_t(df)$$

$$dP_t = \delta P_t \text{ and } E|T_x F_t|\chi_{t<\xi} < \infty \xrightarrow{A=\frac{1}{2}\Delta^h} E\sup_{s\le t}|TF_s| < \infty + \text{complete}$$

$$\text{Gradient BM}+\sup_{s\le t}\sup_{x\in M} E\left(|TF_s|^{2+\delta}\chi_{t<\xi}\right) < \infty \xrightarrow{\text{chap 7}} E\sup_{s\le t}|TF_s| < \infty + \text{complete}$$

$$\int_0^\infty \sup_{x\in K} E(|TF_s|\chi_{s<\xi})ds < \infty + dP_t = \delta P_t \xrightarrow{\text{chap 7}} \text{finite h-volume}$$

$$\begin{matrix}\int_0^\infty \sup_{x\in K} E|TF_s|ds < \infty \\ \sup_{x\in K} E|TF_s|^{1+\delta} < \infty\end{matrix} + \text{strong 1-complete} \xrightarrow{\text{chap 7}} \pi_1(M) = \{0\}$$

$$E(\sup_{s\le t}|TF_s|^{p+\delta}) < \infty + \text{complete} \xrightarrow{\text{chap 5}} \text{strong p-completeness}$$

$$\text{strong p-completeness} \xrightarrow[\text{+ strong } p^{th} \text{ moment stable}]{\text{chap 7}} \text{bounded closed p-forms are exact}$$

$$\begin{matrix}\int_0^t E|TF_s|^2 ds < \infty \\ dP_t f = \delta P_t(df) \\ \text{complete}\end{matrix} \xrightarrow{\text{chapter 8}} \nabla \log p_t^h(\cdot, y_0)(x_0) = E\left\{\frac{1}{t}\int_0^t (TF_s)^*(X dB_s)|x_t = y_0\right\}$$

The arrows here denote implications in the indicated chapter.

1.2 Probabilistic set-up

Brownian motions and Brownian systems

A path continuous strong Markov process on a Riemannian manifold is called a *Brownian motion* (BM) if its generator is $\frac{1}{2}\triangle$, where $\triangle$ denotes the Laplace-Beltrami operator. It is called a Brownian motion with drift Z if its generator is $\frac{1}{2}\triangle + Z$. Here Z is a C^∞ smooth vector field on M. The drift Z is called a *gradient drift* if $Z = \nabla h$ for some function $h : M \to R$. We will always assume h is C^∞ smooth for simplicity. But in many cases we only need it to be C^2.

A stochastic dynamical system (X, A) is called a *Brownian system* (with drift Z) if its solution is a Brownian motion (with drift Z). Equivalently $X(x) : R^m \to T_xM$ is an orthogonal projection for each $x \in M$. i.e. $X^*X = Id$ and $Z = A^X =: A + \frac{1}{2}\text{trace}\nabla X(X(\cdot))(\cdot)$.

In the text, we often use *h-Brownian system* for a Brownian system with drift ∇h, and *h-Brownian motion* for a Brownian motion with drift ∇h.

Gradient Brownian flow

Let $f : M \to R^m$ be an isometric immersion. Define $X(x) : R^m \to T_xM$ as follows:

$$X(x)(e) = \nabla < f(\cdot), e > (x), \quad e \in R^m.$$

A stochastic dynamical system (X, A) with X so defined is called a *gradient Brownian system* (with drift). When $A = 0$, its solution flow is called a *gradient Brownian flow*. It is called a *h-gradient Brownian system* if $A = \nabla h$. For such a system, we can always choose, for each $x \in M$, an orthonormal basis (o.n.b.) $\{e_1, \ldots, e_m\}$ of R^m, such that for all $v \in T_xM$ and $i = 1, \ldots, m$, either

$$\nabla X(v)(e_i) = 0$$

or

$$X(x)(e_i) = 0.$$

Here $\nabla X(-)(e_i)$ denotes for the covariant derivative of $X(e_i)$. In particular we have

$$\sum_{i=1}^{m} \nabla X^i(X^i) = 0. \tag{1.2}$$

Here $X^i(x) = X(x)(e_i)$. See [31] or [25] for the proof.

The derivative flow

Let $M_t(x) = \{\omega : t < \xi(x,\omega)\}$. Denote by TF_t the solution flow to the following covariant stochastic differential equation:

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)dt. \tag{1.3}$$

It is in fact the derivative of F_t in measure in the following sense as shown in [31]: let $f : M \to R$ be a C^1 map and $\sigma: [0,1] \to M$ a smooth curve with $\sigma(0) = x$, and $\dot\sigma(0) = v$. Then for any $\delta > 0$:

$$\lim_{r \to 0} P\{\omega \in M_t(x) : |\frac{f(F_t(\sigma(r)) - f(F_t(x))}{r} - df(TF_t(v))| > \delta\} = 0.$$

By convention $f(F_t(\sigma(r)),\omega) = 0$ if $\omega \notin M_t(\sigma(r))$. Let $x_0 \in M$, and let $v_0 \in T_{x_0}M$. We write $x_t = F_t(x_0)$, $v_t = TF_t(v_0)$ for simplicity. Clearly $TF.$ is a solution to the stochastic differential equation on the tangent bundle TM corresponding to equation 1.3. Furthermore it has the same explosion time as $x.$ according to [26]. In a trivialisation we may not distinguish between the derivative flow and its principal part, if there is no confusion caused.

Itô formula for forms

First, we recall *Itô formula for one forms* from [26]: Let T be a stopping time with $T < \xi$, then

$$\begin{aligned}\phi(v_{t\wedge T}) = \ & \phi(v_0) + \int_0^{t\wedge T} \nabla\phi\,(X(x_s)dB_s)\,(v_s) \\ & + \int_0^{t\wedge T} \phi\,(\nabla X(v_s)dB_s) + \int_0^{t\wedge T} \mathcal{L}\phi(v_s)\,ds.\end{aligned} \tag{1.4}$$

Here $\mathcal{L}$ is the differential operator on 1-forms associated with a s.d.s. (X, A) defined as follows (for Levi-Civita connection):

$$\begin{aligned}(\mathcal{L}\phi)_x(v) = \ & \tfrac{1}{2}\mathrm{trace}\nabla^2\phi(X(x)(\cdot), X(x)(\cdot))(v) \\ & +\tfrac{1}{2}\phi(\mathrm{trace}R(X(x)(\cdot), v)X(x)(\cdot)) \\ & +L_A x + \mathrm{trace}\nabla\phi(X(x)(\cdot))\nabla X(v)(\cdot).\end{aligned} \tag{1.5}$$

If $\mathcal{A} = \frac{1}{2}\triangle + L_Z$ and ϕ is closed, then as shown in [26]:

$$\mathcal{L}\phi = \frac{1}{2}\triangle^1 + L_Z.$$

For higher order forms, there is a similar formula from [26]; we quote here the formula for gradient systems. But first we need some notation (see [1]):

Let A be a linear map from a vector space E to E. We define $(d\Lambda)^q A$ from $E \times \ldots \times E$ to $E \times \ldots \times E$ as follows:

$$(d\Lambda)^q A(v^1, \ldots, v^q) = \sum_{j=1}^{q}(v^1, \ldots, Av^j, \ldots, v^q).$$

Let $v_0 = (v_0^1, \ldots, v_0^q)$, where $v_0^i \in T_{x_0}M$. Denote by v_t the q vector induced by TF_t:

$$v_t = (TF_t(v_0^1), TF_t(v_0^2), \ldots, TF_t(v_0^q)).$$

Here is *Itô formula for gradient Brownian systems for q forms* as given in [26]:

$$\begin{aligned} \psi(v_t) = \; & \psi(v_0) + \int_0^t \nabla\psi(X(x_s)dB_s)(v_s) \\ & + \int_0^t \psi\left((d\Lambda)^q(\nabla X(\cdot)(dB_s)(v_s)\right) + \int_0^t \tfrac{1}{2}\triangle^{h,q}(\psi)(v_s)ds. \end{aligned} \tag{1.6}$$

Moment exponents: For $p \in R$, $K \subset M$ compact, we have *moment exponents* $\mu_K(p)$:

$$\mu_K(p) = \overline{\lim_{t\to\infty}} \sup_{x\in K} \frac{1}{t} \log E|T_xF_t|^p$$

and point moment exponents $\mu_x(p)$:

$$\mu_x(p) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log E|T_xF_t|^p.$$

We will say a flow is *moment stable* if $\mu_x(1) < 0$ for each x, and *strongly moment stable* if $\mu_K(1) < 0$ for each compact set K. It is called *strongly p^{th}-moment stable* if $\mu_K(p) < 0$ for each compact subset K of M. For discussions on various exponents and related problems, see Arnold[2], Baxendale and Stroock [7], Carverhill, Chappel and Elworthy [11], Chappel[12], and Elworthy [26] [25].

Invariant measure: A σ-finite measure m on M is called an invariant measure for $F_\cdot$ if the following holds for all $t > 0$ and L^1 functions f:

$$\int_M P_t f(x) m(dx) = \int_M f(x) m(dx).$$

Some notation: Let dx be the Riemannian volume element on M. Then M is said to have finite volume if $\mathrm{Vol}(M) = \int_M dx < \infty$. Similarly M is said to have *finite h-volume* if h-Vol$(M) = \int_M e^h\, dx < \infty$. e.g. R^n has finite volume for e^h the Gaussian density given by $h(x) = -\frac{1}{2}|x|^2$.

1.3 Unbounded operators on Hilbert space

Let $\mathcal{H}$ be a Hilbert space. An operator T on $\mathcal{H}$ is a linear map from a subspace of $\mathcal{H}$ to $\mathcal{H}$. The subspace is called its domain, which we denote by $D(T)$. We also denote by $\ker(T)$ the kernel of T, and $Im(T)$ (or $\mathrm{Ran}(T)$) the image of T in $\mathcal{H}$. A *closed* operator T is an operator with its graph $\Gamma(T) = \{(\phi, T\phi) : \phi \in D(T)\}$ a closed subspace of $\mathcal{H} \times \mathcal{H}$. An operator S is called an *extension* of T if $\Gamma(S) \supset \Gamma(T)$, i.e. $D(S) \supset D(T)$ and $S = T$ on $D(T)$. If T has a closed extension, it is *closable*. In this case the smallest extension of it is called its *closure*, which we denote as $\bar{T}$.

Proposition 1.3.1 *[59][p.250]. If T is closable, $\Gamma(\bar{T}) = \overline{\Gamma(T)}$, i.e. $D(\bar{T})$ contains precisely those ϕ such that: there exist $\{\phi_n\} \subset D(T)$ with $\phi_n \to \phi$ in $\mathcal{H}$ and $T\phi_n \to \eta$, in $\mathcal{H}$, some $\eta \in \mathcal{H}$ (and then $\bar{T}\phi = \eta$).*

An interesting class of operators are operators with dense domain, called densely defined operators. Let T be such an operator, there is a well defined adjoint operator T^* defined by:

$$D(T^*) = \{\phi \in \mathcal{H} :< T\psi, \phi >=< \psi, \eta >, \text{ some } \eta \in \mathcal{H}, \text{ all } \psi \in D(T)\}.$$

For such ϕ and η we set: $T^*\phi = \eta$.

Notice that if $S \subset T$, then $T^* \supset S^*$. If $T \subset T^*$, T is called a *symmetric operator*. There are also the following properties of adjoint operators:

Theorem 1.3.2 *[59][p.253]. Assume $D(T)$ is dense in $\mathcal{H}$, then:*

1. *T^* is closed, and T^{**} is symmetric.*

2. *T is closable if and only if $D(T^*)$ is dense, in which case $\bar{T} = T^{**}$.*

3. *If T is closable , then $(\bar{T})^* = T^*$.*

A symmetric operator is *self-adjoint* if $T = T^*$. It is *essentially self-adjont* if its closure is self-adjoint, equivalently if it has only one self-adjoint extension.

Here is another test for essential self-adjointness:

Theorem 1.3.3 *[59] [p.257]. Let T be a symmetric operator on a Hilbert space. Then the following are equivalent:*

1. *T is essentially self-adjoint.*
2. *$ker(T^* \pm i) = \{0\}$.*
3. *$Ran(T \pm i)$ are dense.*

The Friedrichs extension: Let T be a positive symmetric operator. Let ε be its quadratic form (bilinear form) defined as follows:

$$\varepsilon(\phi, \psi) =: <\phi, T\psi>, \qquad \text{for } \phi, \psi \in D(T).$$

Let $|\phi|_\varepsilon^2 = |\phi|^2 + \varepsilon(\phi, \phi)$. A quadratic form is *closed* if $\{\phi_n\}$ is a sequence in $D(T)$ with $\lim_{n\to\infty} \phi_n = \phi$ and $lim_{n\to\infty}\varepsilon(\phi_n, \phi_m) = 0$, then it follows $\phi \in D(T)$ and ϕ_n converges to ϕ in $||_\varepsilon$ norm. The *closure* of ε is the least extension of ε which is closed. The closure of ε is in fact the quadratic form of a unique self adjoint operator $\tilde{T}$, which is a positive extension of T and is called the Friedrichs extension of T.

Theorem 1.3.4 *(Von Neumann theorem)[69] Let T be a closed densely defined operator, then T^*T and TT^* are self-adjoint.*

Furthermore, If T is symmetric and T^2 is densely defined, then T^*T is the Friedrichs extension of T^2. See [59] [p.181]

Given operators S and T, we may define new operators $S + T$, and ST. For this, $S + T$ is defined on $D(S) \cap D(T)$, on which $(S + T)\phi =: S\phi + T\phi$, while $D(ST) = \{\phi : \phi \in D(T), T\phi \in D(S)\}$, and $(ST)\phi =: S(T\phi)$.

Theorem 1.3.5 (spectral theorem-functional calculus form) *[59]*

Let A be a self-adjoint operator on $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on R into $L(\mathcal{H})$ so that

1. *$\hat{\phi}$ is an algebraic *-homomorphism.*

2. *$\hat{\phi}$ is norm continuous, that is, $||\hat{\phi}(h)||_{L(\mathcal{H})} \leq ||h||_\infty$.*

3. *Let $\{h_n\}$ be a sequence of bounded Borel functions with $\lim_{n\to\infty} h_n(x)$ converging to x for each x and $|h_n(x)| \leq |x|$ for all x and n. Then, for any $\psi \in D(A)$, $\lim_{n\to\infty} \hat{\phi}(h_n)\psi = A\psi$.*

4. *If $h_n(x) \to h(x)$ pointwise and if the sequence $||h_n||_\infty$ is bounded, then $\hat{\phi}(h_n) \to \hat{\phi}(h)$ strongly.*

5. *If $A\psi = \lambda\psi$, $\hat{\phi}(h)\psi = h(\lambda)\psi$.*

6. *If $h \geq 0$, then $\hat{\phi}(h) \geq 0$.*

1.4 Semigroups and Generators

Definition 1.4.1 *A family of operators $\{T_t\}$ on a Banach space $(X, |\cdot|)$ is called a one parameter semigroup (of class C_0) if they satisfy the following: (not to be confused with the C_0 semigroup later in chapter 4.)*

1. *$T_{t+s} = T_t T_s, t \geq 0, s \geq 0$.*

2. *$T_0 = I$.*

3. *For each $t_0 \geq 0$, $f \in X$,*

$$\lim_{t\to t_0} |T_t f - T_{t_0} f| = 0.$$

For such a semigroup, all T_t are bounded operators with $|T_t| \leq Me^{\beta t}$, for $0 \leq t < \infty$ and some constants $M > 0$, $\beta < \infty$.

The third condition in the definition is equivalent to each of the following if T_t satisfies the semigroup property (1) and (2) in the definition above:[17], [59]

1. It is weakly continuous at 0: $w - \lim_{t\to 0} T_t f = f$, each $f \in X$. i.e. for any $\phi \in X^*$, the dual space, $\lim_{t\to 0} \phi(T_t f) = \phi(f)$.

2. The map $(t, f) \mapsto T_t f$ from $[0, \infty) \times X \to X$ is jointly continuous.

Denote by $D(\mathcal{A}) = \{f : \lim_{t\to 0} \frac{T_t f - f}{t} \text{ exists}\}$. Let $\mathcal{A}f = \lim_{t\to 0} \frac{T_t f - f}{t}$, if $f \in D(\mathcal{A})$. The operator $\mathcal{A}$ on X is called the *infinitesimal generator*, which enjoys the following properties:

Theorem 1.4.1 *[17] Let T_t be a semigroup of class C_0, $\mathcal{A}$ its generator. Then the following hold:*

1. *The operator $\mathcal{A}$ is closed and densely defined.*

2. *$T_t\{D(\mathcal{A})\} \subset D(\mathcal{A})$.*

3. *If $f \in D(\mathcal{A})$, $T_t f$ is continuously differentiable on $[0, \infty)$ and satisfies:*

$$\frac{\partial(T_t f)}{\partial t} = \mathcal{A}(T_t f) = T_t(\mathcal{A}f).$$

4. *Furthermore if a function $g : [0, a] \to Dom(\mathcal{A})$ satisifies*

$$\frac{\partial g_t}{\partial t} = \mathcal{A}g_t$$

for all $t \in [0, a]$, then $g_a = T_a g_0$.

Definition 1.4.2 *A contraction semigroup is a semigroup of class C_0 with $|T_t| \leq 1$ for all $t \geq 0$.*

Theorem 1.4.2 (The Reisz-Thorin interpolation theorem) *[18]*

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let T be a linear operator from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} + L^{q_1}$ which satisfies:

$$|Tf|_{q_i} \leq M_i |f|_{p_i}$$

for all f and $i = 1, 2$. Let $0 < t < 1$ and define p, q by:

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_0},$$

$$\frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0}.$$

Then $|T_t f|_q \leq (M_1)^t \cdot (M_0)^{1-t} |f|_p$ for all $f \in L^{p_0} \cap L^{p_1}$.

In particular if T is a bounded operator both on L_2 and L_∞: then T extends to a operator on L_p, for $2 < p < \infty$. Furthermore if T is both an L_2 and L_∞ contraction, it is an L_p contraction for all such p (restricting to L^2 functions).

Finally we quote the following theorem on dual semigroups from [17]:

Theorem 1.4.3 *[17] If $\mathcal{A}$ is a generator of a one parameter semigroup P_t on a reflexive Banach space X, then P_t^* is a one parameter semigroup on X^* with generator $\mathcal{A}^*$.*

1.5 Differential forms

Let M be a Riemannian manifold with Levi-Civita connection ∇ and a positive measure μ given by $e^{h(x)}dx$, for a smooth function h on M, where dx denotes the volume element determined by the metric. Denote by A^p the space of differentiable p forms. Let ϕ be a p-form; there is an element $\phi^\#(x) \in \wedge^p T_x M$ such that for any $x \in M$, and $v \in \wedge^p T_x M$, $\phi(x)(v) = \, < \phi^\#, v >_x$. For two p-forms ϕ and ψ, we define:

$$< \phi, \psi >_x = < \phi^\#(x), \psi^\#(x) >_x,$$

$$< \phi, \psi > = \int_M < \phi, \psi >_x e^{h(x)} dx.$$

Let $L^2(A^p, \mu)$ be the completion of $\{\phi \in A^p : |\phi|^2 = < \phi, \phi > < \infty\}$, and $L^2(A) = \bigoplus L^2(A^p)$, where $\bigoplus$ denotes direct sums. Then $L^2(A)$ is a Hilbert space with the inner product defined above (forms of different order are considered orthogonal). Those forms in L^2 are called L^2 forms. Let $C_K^\infty(A^p)$ be the space of smooth p forms with compact support, and C_K^∞ the space of smooth functions with compact support. There is the usual exterior differential operator d on C_K^∞ with adjoint δ^h, here δ^h is the formal adjoint of d for μ:

$$d : C_K^\infty(A^p) \to C_K^\infty(A^{p+1}),$$
$$\delta^h : C_K^\infty(A^p) \to C_K^\infty(A^{p-1}).$$

Denote by d^* the adjoint of d in $L^2(A^p, \mu)$ and δ^* the L^2 adjoint of operator δ, so that $d^* = \delta^h$, $(\delta^h)^* = d$ when restricted to C_K^∞. Notice that C_K^∞ is dense in L^2, so all the operators concerned have dense domain.

Denote by $(\wedge^p T_x M)^*$ the dual space of the antisymmetric tensor tangent bundle of order p on R^1. Let $(\wedge T_x M)^* = \bigoplus(\wedge^p T_x M)^*$. For each $e \in T_x M$, there is the annihilation operator $i_e : (\wedge^{p+1} T_x M)^* \to (\wedge^p T_x M)^*$ given by:

$$i_e(\phi)(v_1, \ldots, v_p) = \phi(e_1, v_1, \ldots, v_p).$$

Here we have identified $(\wedge^p T_x M)^*$ with antisymmetric multilinear forms.

Let Y be a vector field on M with S_t the corresponding solution flow. There is the interior product of ϕ by Y given by:

$$i_Y \phi(v_1, \ldots, v_{p-1}) = \phi(Y(x), v_1, \ldots, v_{p-1})$$

for $v_i \in T_x M$. If ψ is a 1-form, we write $i_\psi \phi$ for $i_{\psi^\#}\phi$. Here $\#$ denotes the adjoint. There is also the Lie derivative L_Y of ϕ in direction Y:

$$L_Y \phi(v_1, \ldots, v_p) = \frac{d}{dt}(TS_t(v_1), \ldots, TS_t(v_p))|_{t=0}.$$

Here $v_i \in T_xM$. They are related by the following formula [1]:

$$L_Y\phi = d(i_Y\phi) + i_Y(d\phi).$$

Take an orthonormal basis $e_1, \ldots, e_n$ of T_xM, the formal adjoint δ of the exterior differential operator on $L^2(M, dx)$ is given by:

$$(\delta\phi)_x = -\sum_1^n i_{e_k}(\nabla_{e_k}\phi).$$

For a function f on M,

$$\delta(f\phi) = f\delta\phi - i_{\nabla f}\phi.$$

Also for $\phi \in D(\delta)$, $\psi \in C_K^\infty$,

$$\int < d\phi, \psi > e^h\, dx = \int < \phi, \delta(e^h\psi) > \, dx = \int < \phi, \delta\psi - i_{\nabla h}\psi > e^h\, dx.$$

Thus $\delta^h = \delta - i_{\nabla h}$. Similar arguments show that: $\delta^h = e^{-h}\delta e^h$ and

$$\delta^h(f\phi) = f\delta^h\phi - i_{\nabla f}\phi. \tag{1.7}$$

Let $\triangle$ be the *Hodge-Laplace operator* on forms defined as:

$$\triangle = -(d+\delta)^2,$$

and $\triangle^h$ the *Bismut-Witten Laplacian* defined by:

$$\triangle^h = -(d+\delta^h)^2.$$

Clearly $\triangle^h = \triangle + 2L_{\nabla h}$. The restricted operators on q forms are denote by $\triangle^q$ and $\triangle^{h,q}$ respectively. If $q = 0$, we simply write $\triangle$ and $\triangle^h$.

Divergence theorem Let Y be a vector field. Define the divergence of Y to be: $\mathrm{div}Y = \mathrm{trace}\nabla Y$. It is the formal adjoint of $-\nabla$ on $L^2(M, dx)$. The h-divergence of Y is given by:

$$\text{div}_h Y = e^{-h}\text{div}(e^h Y). \tag{1.8}$$

This is the formal adjoint of $-\nabla$ in $L^2(M, e^h dx)$:

$$\int f(\text{div}_h Y) e^h \, dx = -\int < Y, \nabla f > e^h \, dx.$$

The divergence theorem holds for h-divergences, i.e. let U be a precompact open set in M with piecewise smooth boundary, then:

$$\int_U \text{div}_h Y e^h \, dx = \int_{\partial U} < Y, \nu > e^h \, dS. \tag{1.9}$$

Where ν is the unit outer normal vector to ∂U, and dS the Riemannian surface area element(corresponding to dx). The equation can be proved from equation 1.8 and from the usual divergence theorem with $h = 0$.

1.6 Parabolic regularity etc.

Recall that h is a smooth function on M. Let $P_t^h \phi$ be a L^2 solution to the following equation on q forms:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) &= \left(\frac{1}{2}\Delta^h\right) u(x,t) \\ u(x,0) &= \phi \end{cases}$$

Then $P_t^h \phi$ is in fact a classical solution(i.e. C^2 in x and C^1 in t). Furthermore it is smooth if ϕ and h are. See [56] and [24].

The following theorem is given by Strichartz for the Laplacian, but it is easy to see it is true for $\frac{1}{2}\Delta^h$ (defined on page 21, c.f. theorem 2.1.4). Let $e^{\frac{1}{2}t\Delta^h}$ be the heat semigroup defined by functional analysis.

Theorem 1.6.1 *[62] There is a heat kernel $p_t^h(x,y)$ satisfying:*

1. *The function $p_t^h(x,y)$ is smooth on $R^+ \times M \times M$, symmetric in x and y, and is strictly positive for $t > 0$.*

2. $\int_M p_t^h(x,y)e^h dx \leq 1$ *for all* x *and* $t > 0$, *and*

$$e^{\frac{1}{2}t\Delta^h} f(x) = \int p_t^h(x,y) f(y) e^{h(y)}\, dy \tag{1.10}$$

for all $f \in L_2$.

3. $|e^{\frac{1}{2}t\Delta^h} f|_p \leq |f|_p$, *for all* $t > 0$, *and* $f \in L^2 \cap L^p$, $1 \leq p \leq \infty$ *with* $\lim_{t\to 0} |e^{\frac{1}{2}t\Delta^h} f - f|_p = 0$, *for* $p \neq \infty$, *and*

4. $\frac{\partial}{\partial t} e^{\frac{1}{2}t\Delta^h} f = \frac{1}{2}\Delta^h e^{\frac{1}{2}t\Delta^h} f$ *for all* $f \in L^2$ *and* $t > 0$, *and these properties continue to hold for all* $f \in L^p$, $1 \leq p \leq \infty$ *if we define* $e^{\frac{1}{2}t\Delta^h} f$ *by equation 1.10. Moreover we have uniqueness of the semigroup for* $1 < p < \infty$ *in the following sense: if* Q_t *is any strongly continuous contractive semigroup on* L^p *for fixed* p *such that* $Q_t f$ *is a solution to* $\frac{\partial}{\partial t} = \frac{1}{2}\Delta^h$, *then* $Q_t f = e^{\frac{1}{2}t\Delta^h} f$.

Chapter 2

On the Bismut-Witten Laplacian and its semigroups

Main result: From the essential self-adjointness of $d + \delta^h$, we get the Hodge decomposition theorem and a result on $dP_t^h = P_t^h d$, which we use extensively later. These results are generally considered well known, but there does not appear to be a suitable reference which gives these in exactly the context we want, especially for the Bismut-Witten Laplacian $\triangle^h$. This problem arose when I was trying to work out $dP_t^h = P_t^h d$, and occurred again when D. Elworthy and S. Rosenberg were working on their paper [35]. This chapter is the result of discussions among the three of us, part of the result is in fact used in [35].

2.1 The self-adjointness of weighted Laplacians

Chernoff [14] has proved that $d + \delta^h$ and all its powers are essentially self-adjoint on C_K^∞ (with $h = 0$, but exactly the same proof gives the result for $h \neq 0$ making use of the h-divergence theorem stated on page 22 as seen later in the context). In particular the Hodge-Laplacian operator $\triangle^h = -(d+\delta^h)^2$ is essentially self-adjoint. From this we obtain $\overline{d+\delta^h} = \bar{d}+\overline{\delta^h}$ and the Hodge-Kodaira decomposition, and therefore : $d^* = \delta^h$ and $(\delta^h)^* = d$. Consequently we get: $\overline{\triangle^h} = -(\bar{d}\overline{\delta^h} + \overline{\delta^h}\bar{d})$. Thus all the operations on $\triangle^h$ are exactly as for $\triangle$. We will give Chernoff's proof for the self-adjointness, adapted to our context.

First we introduce the terminology. Let L be a first order differential operator on the complexified cotangent bundle $(\wedge T_x M)^*_C$. Consider $\frac{\partial}{\partial t} = L$, which is called a symmetric hyperbolic system if $L^* + L$ is a zero order operator, i.e. it is given locally by multiplication by a matrix valued function of x. The symbol σ for L is a linear map from $(\wedge T_x M)^*_C \to (\wedge T_x M)^*_C$ for each $x \in M$ and $v \in T^*_x M$:

$$\sigma(v)(e) = L(f\phi)(x) - fL\phi(x). \tag{2.1}$$

Here $f \in C^\infty(M)$ with $df(x) = v$, and $\phi : M \to (\wedge TM)^*$ is a smooth form on M with $\phi(x) = e$. The local velocity of propagation c associated to L is defined as a function on M:

$$c(x) = \sup_v\{||\sigma(v)(x)|| : v \in T^*_x M, ||v|| = 1\}.$$

We are mainly interested in the following operator $L = i(d+\delta^h)$ for the proof of the next two theorems. It indeed gives rise to a hyperbolic system since $L + L^* = 0$, and has symbol: $\sigma(v)(e) = v \wedge e - i_{v^\#}e$. Here $v^\#$ is defined by: $v(w) =< v^\#, w >$ for $w \in T_x M$. The local velocity for L is constantly 1, from the following:

For each fixed $v \neq 0$, we may write e into the sum of two orthogonal parts: $e = e_0 + e_1$ satisfying: $v \wedge e_0 = 0$, and $|v \wedge e_1| = |v||e_1|$, by letting e_0 be the component of e along v. So $i_{v^\#}e_1 = 0$. Thus

$$|\sigma(v)| = \sup_{|e|=1}(|v \wedge e - i_{v^\#}e|) = \sup_{|e|=1}(|v \wedge e_1| + |i_{v^\#}e_0|) = |v|.$$

Lemma 2.1.1 (Energy inequality) *[14] Let M be a Riemannian manifold with Riemannian distance ρ. The geodesic ball radius R centered at x_0 is denoted by $B(x_0, R)$. Let $x_0 \in M$, u a smooth solution to $\frac{\partial}{\partial t} = L$ on $[0, a] \times B(x_0, R)$. Let r and a be real numbers with $r + a \leq R$, then the following inequality holds:*

$$\int_{B(x_0,r)} |u(a)|^2 e^h \, dx \leq \int_{B(x_0,r+a)} |u(0)|^2 e^h \, dx.$$

In particular if $u(0)$ vanishes on $B(x_0, r+a)$, then u vanishes throughout K for $K = \{(t, x) : 0 \leq t \leq a, \rho(x, x_0) \leq r + (a - t)\}$.

Proof: We first introduce a (h-)divergence free vector field Z on $[0, a] \times M$ as follows. Let f be a real valued function on $[0, a] \times M$, denote by $\frac{\partial f}{\partial t}$ the differential of f in the time component, and df the space differential. Then define Z by:

$$df(Z) = < Z, \nabla f > = |u_t|^2 \frac{\partial f}{\partial t} - < u_t, \sigma(df) u_t > .$$

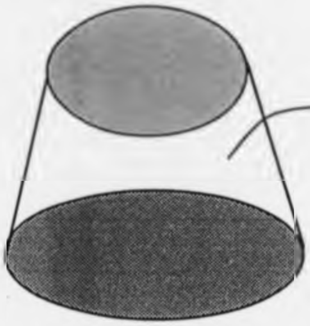
Write $Y(f) = < u_t, \sigma(df) u_t >$. Notice Z is given in the form of the sum of time and space parts, so therefore is the divergence:

$$\mathrm{div}_h Z = 2 < u_t, \frac{\partial}{\partial t} u_t > - \tilde{\mathrm{div}}_h Y,$$

where $\tilde{\mathrm{div}}_h Y$ denotes the h-divergence of Y in its space variables.

But $\tilde{\mathrm{div}}_h Y = < (L - L^*) u_t, u_t >$, since for any function $f \in C_K^\infty$:

$$\begin{aligned} \int_M f \tilde{\mathrm{div}}_h Y \, e^h dx &= -\int_M < Y, \nabla f > \, e^h dx \\ &= -\int < u_t, L(f u_t) - f L u_t > \, e^h dx \\ &= -\int_M \{ < L^* u_t, u_t > - < u_t, L u_t > \} f \, e^h dx \end{aligned}$$



$$= \; -\int_M f < u_t, (L^* - L)u_t > e^h dx.$$

Thus $\mathrm{div}_h Z =< u_t, (L + L^*)u_t >= 0$. Applying divergence theorem for the h-divergence, we get:

$$\begin{aligned} 0 &= \int_K \mathrm{div}_h Z \, e^h dt dx \\ &= \int_{\partial K} < Z, \nu > e^h dS \\ &= \int_{B(x_0,r)} |u(a)|^2 \, e^h dx - \int_{B(x_0,r+a)} |u(0)|^2 \, e^h dx + \int_\Sigma < Z, \nu > e^h dS. \end{aligned}$$

Here Σ is the mantle part of the boundary of K, and ν the unit outer normal vector to Σ, which is in fact given by: $\nu = \alpha \nabla \phi = \alpha(1, \nabla \rho)$, for $\phi(t, x) = t + \rho(x_0, x)$, and some positive constant α.

$$\begin{aligned} < Z, \nu > &= \alpha |u_t|^2 - < u_t, \sigma(\alpha \, d\rho) u_t > \\ &\geq \alpha |u_t|^2 - \alpha |d\rho| |u_t|^2 = 0. \end{aligned}$$

The required inequality follows. ■

Let $L = i(d + \delta^h)$. We shall view the equation $\frac{\partial u}{\partial t} = i(d + \delta^h)u$ as a symmetric hyperbolic system of partial differential equations as follows:

Let $\{\omega_1, \ldots, \omega_{2n}\}$ be local sections of $A = \bigoplus_{p=0}^n A_p$ which are pointwise orthonormal. Then we may wite $v = \sum_1^n v_\alpha(x)\omega_\alpha$ and

$$(d+\delta^h)v = \sum_{i,\alpha,\beta} A_{i\alpha\beta}(x)\frac{\partial v_\alpha}{\partial x_i}\omega_\beta + \sum_{\alpha,\beta} B_{\alpha\beta}v_\alpha(x)\omega_\beta.$$

Let $v, w \in C^\infty(A)$, the space of smooth forms in A. We shall assume that one of them has compact support. The identity

$$< (d+\delta^h)v, w > = < v, (d+\delta^h)w >$$

and an integration by parts show that

$$A_{i\alpha\beta}(x) = -A_{i\beta\alpha}(x), \text{ any } i, \alpha, \beta, \text{ and } x.$$

We shall let $A_i(x)$ denote the matrix whose entries are $A_{i\alpha\beta}(x)$. On setting $u = v + iw$, the equation: $\frac{\partial u}{\partial t} = i(d+\delta^h)u$ can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} = \sum_1^n \bar{A}_i(x)\frac{\partial}{\partial x^i}\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} + \bar{B}(x)\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix},$$

where $\vec{v} = (v_1, \ldots, v_{2n})$ and $\vec{w} = (w_1, \ldots, w_{2n})$ and $\bar{A}_i(x) = \begin{pmatrix} 0 & A_i(x) \\ -A_i(x) & 0 \end{pmatrix}$ is symmetric.

Thus locally there is a unique smooth solution to the Cauchy problem for this hyperbolic equation by standard theory. See e.g. [48].

Theorem 2.1.2 *[14] Let M be a complete Riemannian manifold, and let $L = \sqrt{-1}(d+\delta^h)$. There is a unique smooth solution to the hyperbolic equation: $\frac{\partial}{\partial t} = L$, for each initial value $u_0 \in C_K^\infty$. Moreover $u_t \in C_K^\infty$, each t.*

Proof: Take a local trivialization of the tangent bundle TM. For $y \in M$, there is a constant $r(y) > 0$ such that $B(y, r)$ is contained in a single chart. Then by the local existence and uniqueness theorem: for each $u(0)$ smooth on $B(y, r)$, there is a unique smooth solution on $K_y = \{(t, x) : 0 \le t \le$

$r, \rho(x,y) \leq r - t\}$, from the energy inequality. Notice the propagation speed is identically 1 for L.

For $R > 0$, $x_0 \in M$, the geodesic ball $B(x_0, R)$ is compact on a complete Riemannian manifold. Let $u(0)$ be smooth on $B(x_0, R)$. We can find a common $r > 0$ for all points in the ball $B(x_0, R - r)$ such that the statement above holds, given initial data on $B(x_0, R)$. Moreover the solutions coincide on overlapping areas from the local uniqueness. Altogether they define a solution on the truncated cone:

$$\{(t,x) : 0 \leq t \leq r, \rho(x, x_0) \leq R - t\}.$$

The solution at $t = r$ in turn serves as initial data on $B(x_0, R - r)$, then we have a solution on: $\{r \leq t \leq 2r, \rho(x, x_0) \leq R - r - t\}$. Continuing with the procedure, we obtain a unique solution $\{(t,x) : 0 \leq t \leq 1, \rho(x, x_0) \leq R - t\}$.

In particular this show that the solution up to time N on $B(x_0, R - N)$ is determined by solution at time $N - 1$ on the ball $B(x_0, R - N + 1)$, and therefore by initial value on $B(x_0, R)$.

Let $u(0) \equiv 0$ on M. For each number N and R, $u \equiv 0$ on $[0, N] \times B(x_0, R)$ from the above argument. So the uniqueness holds.

We show next that there is a globally defined solution for each smooth data u_0 with compact support inside the ball $B(x_0, a)$, some a. As has been shown there is a unique solution on $K = [0,1] \times B(x_0, a + 3 - t)$. Moreover the solution vanishes outside $B(x_0, a + 1)$. Extend the solution to $[0,1] \times M$ by setting it to be 0 outside K. The solution so defined is smooth and h as support in $B(x_0, a + 1)$ at time $t = 1$, which serves in turn as initial data and gives us a solution on $[0, 2] \times M$. We can by this means propagate the solution for all time. It is clear that the solution u_t thus obtained has compact support for each t. ■

Theorem 2.1.3 *[14] Denote by $V_t u$ the solution given in the theorem above* *for $u \in C_K^\infty$. Then V_t is a unitary group from $C_K^\infty \to C_K^\infty$ with $L(V_t u) = V_t(Lu)$, and extends to L^2 as a unitary group.*

Proof: We only need to prove the unitary part. Take u, w from C_K^∞.Than

$$\begin{aligned} \frac{d}{dt} < V_t u, V_t w > &= < LV_t u, V_t w > + < V_t u, LV_t w > \\ &= < (L + L^*)V_t u, V_t w >= 0. \end{aligned}$$

The fact that V_t is a group comes from the uniqueness of the solution, and $L(V_t u) = V_t(Lu)$ follows from the standard semigroup result. ■

Theorem 2.1.4 *[14] The operator $T = d + \delta^h$ and all its powers are essentially self-adjoint.*

Proof: Let $A = T^n$, $n > 0$. It is a symmetric operator. According to theorem 1.4 we only need to show $\psi = 0$ if $A^*\psi = \pm i\psi$, $i = \sqrt{-1}$. Suppose $A^*\psi = i\psi$. Let $\phi \in C_K^\infty$. Consider $< V_t\phi, \psi >$ which is bounded in t since V_t is a unitary operator.

$$\begin{aligned} \frac{d^n}{dt^n} < V_t\phi, \psi > &= i^n < T^n V_t\phi, \psi > \\ &= i^n < V_t\phi, A^*\psi >= i^n < V_t\phi, i\psi > \\ &= -i^{n+1} < V_t\phi, \psi > . \end{aligned}$$

So $< V_t\phi, \psi >= Ce^{\alpha t}$, for some constant C. Where α satisfies: $\alpha^n + i^{n+1} = 0$, and so is not a pure imaginary. Thus $< V_t\phi, \psi >= 0$, since it is bounded in t. In particular $< \phi, \psi >= 0$. Thus $\psi = 0$. A similar argument works with $A^*\psi = -i\psi$. ■

2.2 The Hodge decomposition theorem

To explore the relationship of these operators, let us first notice that $d \subset \delta^*$, and $\delta \subset d^*$ before giving the following lemmas:

Lemma 2.2.1 *On $L^2(M, e^h dx)$, $D(d^2) = D(d)$, $D((\delta^h)^2) = D(\delta^h)$, and $(d)^2 = (\delta^h)^2 = 0$. It is also true that: $D(d) = D((\delta^h)^* d)$, and $(\delta^h)^* d = 0$. Similarly for $d^* \overline{\delta^h}$.*

Proof: Take $\phi \in D(d)$, there are therefore ϕ_n in C_K^∞ converging to ϕ in L^2, and $d\phi_n \to d\phi$ in L^2. But $d^2\phi_n = 0$, so $d\phi \in D(d)$, and $d^2\phi = 0$.

Let $\phi \in D(d)$, then for all $\psi \in D(\delta^h) = C_K^\infty$, we have:

$$< d\phi, \delta^h\psi > = < \phi, d^*\delta^h\psi > = 0.$$

So $d\phi \in D(\delta^*)$ and $(\delta^h)^*(d\phi) = 0$. The rest can be proved analogously. ■

Proposition 2.2.2 *Let M be a complete Riemannian manifold. Then:*
$\bar{d} + \bar{\delta}^h = \overline{d + \delta^h} = d^* + (\delta^h)^*$.

Proof: We write δ for δ^h in the proof. Since both d^*, and δ^* are closed from theorem 1.2, so is $d^* + \delta^*$ since d^* and δ^* map to different spaces. For the same reason $d + \delta$ is also a closed operator. Noticing $d \subset \delta^*$, $\delta \subset d^*$, we have:

$$\overline{d + \delta} \subset \bar{d} + \bar{\delta} \subset d^* + \delta^*. \tag{2.2}$$

We also have:

$$d^* + \delta^* \subset (d + \delta)^* \tag{2.3}$$

from the following argument:

Let $\phi \in D(d^* + \delta^*)$. For any $\psi \in D(d + \delta)$, there is the following:

$$< (d + \delta)(\psi), \phi > = < \psi, (d)^*\phi > + < \psi, (\delta)^*\phi > = < \psi, (d^* + \delta^*)\phi > .$$

The last equality comes from theorem 1.3.2. We now take adjoint of both sides of 2.2 and 2.3 and get:

$$(d^* + \delta^*)^* \subset (d + \delta)^* \subset (\overline{d + \delta})^*, \tag{2.4}$$

$$d + \delta = (d + \delta)^{**} \subset (d^* + \delta^*)^*. \tag{2.5}$$

But by theorem 2.1.4, $(\overline{d + \delta})^* = \overline{d + \delta}$.

So combine 2.2 with 2.4 and 2.5, we have:

$$\overline{d + \delta} \subset \dot{d} + \dot{\delta} \subset (d^* + \delta^*)^* \subset \overline{d + \delta}.$$

Therefore $\dot{d} + \dot{\delta} = (d^* + \delta^*)^* = \overline{d + \delta}$, and each is a self-adjoint operator. Thus we have proved that:

$$\dot{d} + \dot{\delta} = d^* + \delta^* = \overline{d + \delta}.$$

■

Remark: Let $\overline{\triangle^h} = -\overline{(d + \delta^h)^2}$, it is now clear that:

$$\overline{\triangle^h} = -(\dot{d} + \dot{\delta}^h)^2 = -(\dot{d}\dot{\delta}^h + \dot{\delta}^h \dot{d}).$$

Proof: First we know that $\dot{d} + \dot{\delta}^h$ is self adjoint from 2.1.4. By theorem 1.3.4, $\triangle^h$ is self adjoint. But there is only one self adjoint extension for $\triangle^h$. The result follows. Furthermore $(\dot{d} + \dot{\delta}^h)^2$ is in fact the Friedrichs extension of $\triangle^h$ from the remark after theorem 1.3.4.

The following is a standard result:

Proposition 2.2.3 *Let T be a self-adjoint operator on Hilbert space $\mathcal{H}$ with dense domain. Then*

$$H = \overline{Im(T)} \bigoplus ker(T).$$

Proof: Let $\phi \in D(T)$, $\psi \in ker(T)$. Then

$$< T\phi, \psi > = < \phi, T\psi > = 0.$$

So $\overline{Im(T)}$ is orthogonal to $ker(T)$.

Let $\phi \in (\overline{Im(T)})^{\perp}$. Then by definition for any $\psi \in D(T)$, we have: $<\phi, T\psi> = 0$. However this shows that: $\phi \in D(T^*) = D(T)$, and $T^*\phi = 0$. Thus $T\phi = 0$. Therefore $\overline{Im(T)}^{\perp} \subset Ker(T)$. This finishes the proof. ■

We are ready to prove the following Hodge decomposition theorem:

Theorem 2.2.4 *Let M be a complete Riemannian manifold with measure $e^h dx$, here $h \in C^\infty(M)$. Let $L^2(\mathcal{H}) = \{\phi : d\phi = \delta^h\phi = 0\}$. Then:*

$$L^2\Omega^p = \overline{Im(\delta^h)} \bigoplus \overline{Im(d)} \bigoplus L^2(\mathcal{H})$$

and

$$L^2(\mathcal{H}) = Ker\overline{\Delta^h}.$$

Proof: From the propositions above, we know:

$$L^2 = \overline{Im(d+\delta)} \bigoplus Ker(d+\delta).$$

We only need to prove the following:

$$\overline{Im(d+\delta)} = \overline{Im(d)} \bigoplus \overline{Im(\delta)}.$$

It is clear the two spaces on the right hand side are orthogonal to each other. Take $\phi \in \overline{Im(d+\delta)}$. By definition there are $\phi_n \in D(d+\delta)$, such that $(d+\delta)\phi_n \to \phi$. Since $d\phi$ and $\delta\phi$ are in different spaces, we may write $\phi = \phi_1 + \phi_2$ such that: $d\phi_n \to \phi_1$, and $\delta\phi_n \to \phi_2$. Thus follows:

$$\overline{Im(d+\delta)} \subset \overline{Im(d)} \bigoplus \overline{Im(\delta)}.$$

Next let $\phi \in D(d)$. Let $\psi \in Ker(d+\delta) = Ker(d) \cap Ker(\delta)$. Take $\psi_n \in C_K^\infty$ converging to ψ such that $\delta\psi_n \to \delta\psi = 0$. Then there is the following:

$$< d\phi, \psi > = \lim_{n\to\infty} < d\phi, \psi_n > = \lim_n < \phi, \delta\psi_n > = < \phi, \delta\psi > = 0.$$

Therefore we get: $\overline{Im(d)} \subset ker(\bar{d} + \bar{\delta})^{\perp}$. Similarly $\overline{Im(\delta)} \subset ker(\bar{d} + \bar{\delta})^{\perp}$. Thus

$$L^2\Omega^p = \overline{Im(\delta^h)} \bigoplus \overline{Im(d)} \bigoplus L^2(\mathcal{H}).$$

from $\overline{Im(\bar{\delta}^h)} = \overline{Im(\delta^h)}$, and $\overline{Im(\bar{d})} = \overline{Im(d)}$.

For the proof of the second part, recall $\delta^* d\phi = 0$. For $\phi \in D(\bar{\triangle})$,

$$- < \bar{\triangle}\phi, \phi >=< (\bar{d} + \bar{\delta})\phi, (\bar{d} + \bar{\delta})\phi >= |d\phi|^2 + |\delta\phi|^2.$$

So $L^2(\mathcal{H}) = Ker(\bar{\triangle})$. The proof is complete. ∎

Proposition 2.2.5 *Let M be a complete Riemannian manifold, then:*

$\bar{d} = (\delta^h)^*$, and $\bar{\delta}^h = d^*$.

Proof First notice δ^{h*} is an extension of $\bar{d}$. Let $\phi \in D(\delta^{h*})$, we may write $\phi = \phi_1 + \phi_2 + h$. Here $h \in L^2(\mathcal{H})$, $\phi_1 \in \overline{Im(d)}$, and $\phi_2 \in \overline{Im(\delta^h)}$. However $\overline{Im(d)} \subset D(\bar{d})$, $\overline{Im(\delta^h)} \subset D(\bar{\delta}^h)$. So $h + \phi_1 \in D(\bar{d}) \subset D(\delta^{h^*})$, and ϕ_2 is in the domain of $(\delta^h)^*$. But then $\phi_2 \in D(\delta^{h*}) \cap D(\bar{\delta}^h) \subset D(\delta^{h*}) \cap D(d^*) = D(\bar{d} + \bar{\delta}^h)$, by lemma 1. So $\phi_2 \in D(\bar{d})$, and therefore $\phi \in D(\bar{d})$.

This gives: $D(\delta^{h*}) \subset D(\bar{d})$, i.e. $\delta^* = \bar{d}$. Similarly we can prove $\bar{\delta}^h = d^*$. ∎

Remark: Restricted to C^1 forms, the above lemma can be obtained by an approximation method as suggested by Gaffney [38].

With these established, we are happy to use d, δ^h, and $\triangle^h$ for their closure without causing any confusion.

2.3 The semigroup associated with $\triangle^h$

Since $\triangle^h$ is non-positive, there is a semigroup $e^{\frac{1}{2}t\triangle^h}$ with generator $\frac{1}{2}\triangle^h$ defined by the spectral theorem. Furthermore $e^{\frac{1}{2}t\triangle^h}\alpha$ solves:

$$\begin{cases} \frac{\partial}{\partial t} g_t &= \frac{1}{2}\Delta^h g_t \\ g_0 &= \alpha. \end{cases} \tag{2.6}$$

Here $\alpha \in L^2(\Omega)$ is a L^2 form.

Proposition 2.3.1 *Let M be a complete Riemannian manifold. Let $\alpha \in D(d) \cap D(\delta^h)$, then $de^{\frac{1}{2}t\Delta^h}\alpha = e^{\frac{1}{2}t\Delta^h}(d\alpha)$, and $\delta^h e^{\frac{1}{2}t\Delta^h}\alpha = e^{\frac{1}{2}t\Delta^h}(\delta^h\alpha)$.*

Proof: See Gaffney[39] for a proof of the lemma for C^1 forms (h=0). Let $h(y) = ye^{-\frac{1}{2}ty^2}$, $h_n(y) = \chi_{[-n,n]}(y)(ye^{-\frac{1}{2}ty^2})$, and $g_n(y) = \chi_{[-n,n]}(y)y$. Denote by $h_n(\mathcal{A})$ and $g_n(\mathcal{A})$ the operators defined by the spectral theorem corresponding to an operator $\mathcal{A}$. Then

$$h_n(\mathcal{A}) = g_n(\mathcal{A})e^{-\frac{1}{2}t\mathcal{A}^2} = e^{-\frac{1}{2}t\mathcal{A}^2}g_n(\mathcal{A}).$$

Now $|g_n(y)| \leq |y|$, all n and $g_n(y) \to y$, so the spectral theorem gives the following convergence result:

$$\lim_{n\to\infty} g_n(\mathcal{A})\psi = \mathcal{A}\psi, \quad \psi \in D(\mathcal{A}).$$

Thus for $\psi \in D(\mathcal{A})$, we have ($P_t\psi$ is automatically in $D(\mathcal{A})$):

$$\mathcal{A}e^{-\frac{1}{2}t\mathcal{A}^2}\psi = \lim_{n\to\infty} g_n(\mathcal{A})e^{-\frac{1}{2}t\mathcal{A}^2}\psi = \lim_{n\to\infty} e^{-\frac{1}{2}t\mathcal{A}^2}g_n(\mathcal{A}) = e^{-\frac{1}{2}t\mathcal{A}^2}\mathcal{A}.$$

Now let $\mathcal{A} = d + \delta^h$. Let $\psi \in D(d + \delta^h)$, then:

$$(d + \delta^h)e^{\frac{1}{2}t\Delta^h}\psi = e^{\frac{1}{2}t\Delta^h}(d + \delta^h)\psi.$$

Thus the proof is finished noticing both sides of the equality above consist of orthogonal forms. ■

Chapter 3

Invariant measures and ergodic properties of BM on manifolds of finite h-volume

Let M be a complete Riemannian manifold. There is an invariant measure for h-Brownian motion, i.e. for the diffusion process with generator $\frac{1}{2}\triangle^h$. In this chapter we give a direct proof of the existence of invariant measures and deduce some ergodic properties, which are used in chapter 6 and 7. These ergodic properties are essentially known and well treated in [52] and [61], at least when the weight h is zero. However in their treatment, the processes are required to have the C_0 property if the manifold is not compact. Our treatment avoids the problem of having to assume the C_0 property for the Brownian motion concerned.

The following lemma is a standard result from semigroup theory:

Lemma 3.0.2 *Let M be a complete Riemannian manifold given the measure $e^h dx$. Let $f \in L^2$. Denote by P_t^h the heat semigroup. Then*

$\int_0^t P_s^h f ds \in D(\Delta^h)$ *and moreover:*

$$\frac{1}{2}\bar{\Delta}^h(\int_0^t P_s^h f ds) = P_t^h f - f.$$

To prove this, we only need to show for all $g \in C_K^\infty$:

$$\int_M \left(\int_0^t P_s^h f ds\right)\left(\frac{1}{2}\bar{\Delta}^h g\right) e^h dx = \int_M \left(P_t^h f - f\right) g e^h dx.$$

using the fact that $\frac{1}{2}(\overline{\Delta}^h)^* = \frac{1}{2}\overline{\Delta}^h$. See chapter 2. Noticing the dual semigroup $(P_s^h)^*$ equals P_s^h, we obtain:

$$\begin{aligned}
& \tfrac{1}{2}\textstyle\int_M \left(\int_0^t P_s^h f ds\right)\left(\Delta^h g\right) e^h dx \\
= \;& \tfrac{1}{2}\textstyle\int_0^t \int_M P_s^h f(\Delta^h g)\, e^h dx\, ds, \\
= \;& \tfrac{1}{2}\textstyle\int_0^t \int_M f P_s^h(\Delta^h g)\, e^h dx\, ds, \\
= \;& \tfrac{1}{2}\textstyle\int_0^t \int_M f \bar{\Delta}^h(P_s^h g)\, e^h dx\, ds \\
= \;& \textstyle\int_0^t \int_M f \frac{\partial}{\partial s}(P_s^h g)\, e^h dx\, ds \\
= \;& \textstyle\int_M f \left(\int_0^t \frac{\partial}{\partial s}(P_s^h g)\, ds\right) e^h dx, \\
= \;& \textstyle\int_M f(P_t^h g - g)\, e^h dx \\
= \;& \textstyle\int_M (P_t^h f - f) g\, e^h dx.
\end{aligned}$$

The proof is finished. ∎

Next we notice if M is a complete Riemannian manifold of h-finite volume, then 1 is in the domain of $\bar{\Delta}^h$ (let h_n be the sequence in C_K^∞ approximating the constant function 1 as in the appendix, then $h_n \to 1$ in L^2). Thus we have nonexplosion for a h-Brownian motion in this case as is well known (see [40] for the case of $h = 0$). A quick proof is as follows:

$$\frac{\partial P_t^h 1}{\partial t} = \frac{1}{2}\bar{\Delta}^h(P_t^h 1) = \frac{1}{2}P_t^h(\bar{\Delta}^h 1) = 0.$$

We also have:

Theorem 3.0.3 *Let M be a complete Riemannian manifold, then $e^h dx$ is an invariant measure for P_t^h, i.e.*

$$\int P_t^h f(x) e^h dx = \int f(x) e^h dx \tag{3.1}$$

for each L^1 function f.

Proof: First notice if the invariance property 3.1 holds for functions in C_K^∞, it holds for all L^1 functions since P_t^h is continuous on L^1.

Next there is the following divergence theorem: let ϕ be a L^1 1-form with $\delta^h\phi$ also in L^1, then

$$\int_M \left(\delta^h \phi\right) e^h dx = 0$$

which can be proved by taking approximations of L^1 forms by smooth compactly supported forms and using the Green theorem on page 22, as proved in [42] for $h = 0$.

Let $f \in C_K^\infty$, then $\int_0^t P_s^h f ds$ is in the domain of the h-Laplacian $\triangle^h$ by the previous lemma and we have:

$$\int_M (P_t^h f - f) e^h dx \;=\; \frac{1}{2}\int_M \triangle^h \left(\int_0^t P_s^h f ds\right) e^h dx = 0.$$

■

Recall for an elliptic system our semigroup P_t has the strong Feller property, i.e. it sends $B(M)$ to $C(M)$. See [54]. For such processes there are several notions of recurrence. The basic definition is as follows:

Definition 3.0.1 *Denote by P^x the probability law of a process X starting from x. The process X is called recurrent if for each $x \in M$, the trajectories of X return P^x a.s. infinitely often to any given open set in M. It is called transient if for each $x \in M$, the trajectories X tend P^x a.s. to infinity as $t \to \infty$.*

Let G be the potential kernel of the differential operator $\mathcal{A} = \frac{1}{2}\triangle + L_Z$:

$$Gf(x) = \int_0^\infty P_t f(x) dt. \tag{3.2}$$

Then in potential language, X is transient if and only if G is everywhere finite on compact sets, i.e. when G applied to χ_K for K compact is finite. It is recurrent if and only if G is identically infinite on open sets. So a strong Feller process is either transient or recurrent as proved in [4] following [3].

Theorem 3.0.4 *Let X_t be an h-Brownian motion on a complete Riemannian manifold of finite h-volume, then X_t is recurrent.*

We only need to show that it is not transient. Assume x_t is transient. Let K be a compact set, then $\lim_{t\to\infty} \chi_K(x_t) = 0$ a.s. So $\lim_t E\chi_K(x_t) = 0$ by the dominated convergence theorem and $\int_M E\chi_K(x_t)e^h dx \to 0$. But $\int_M E\chi_K(x_t)e^h dx = \mathrm{vol}(K)$. This gives a contradiction. ■

Proposition 3.0.5 *Let M be a complete Riemannian manifold of h-finite volume. Let μ be the invariant probability measure, then we have for any compact set K:*

$$\lim_{t\to\infty} P_t^h(\chi_K(x)) = \frac{h\text{-}vol(K)}{h\text{-}vol(M)} = \mu(K).$$

Here "vol" denotes the h-volume.

Proof: First notice $\chi_K \in L^2$, so $P_t^h \chi_K$ converges in L^2 to a harmonic function. The convergence is also in L^1 since h-vol$(M) < \infty$, and also the limit function is a constant. So

$$\begin{aligned}\lim_{t\to\infty} \int_M P_t^h \chi_K e^h dx &= \int_M \lim_{t\to\infty} P_t^h \chi_K dx \\ &= (\lim_{t\to\infty} P_t^h \chi_K)\text{h-vol}(M).\end{aligned}$$

But

$$\int_M P_t^h \chi_K dx = \int_M \chi_K dx = \text{h-vol}(K).$$

Thus

$$\lim_{t\to\infty} P_t^h \chi_K = \frac{\text{h-vol}(K)}{\text{h-vol}(M)}.$$

Part II

Completeness and properties at infinity

Chapter 4

Properties at infinity of diffusion semigroups and stochastic flows via weak uniform covers

4.1 Introduction

I. Background

A diffusion process is said to be a C_0 diffusion if its semigroup leaves invariant $C_0(M)$, the space of continuous functions vanishing at infinity, in which case the semigroup is said to have the C_0 property. A Riemannian manifold is said to be stochastically complete if the Brownian motion on it is complete, it is also said to have the C_0 property if the Brownian motion on it does. One example of a Riemannian manifold which is stochastically complete is a complete manifold with finite volume. See Gaffney [40]. More generally a complete Riemannian manifold with Ricci curvature bounded from below is stochastically complete and has the C_0 property as proved by

Yau [68]. See also Ichihara [44], Dodziuk [20], Karp-P. Li [49], Bakry [5], Grigoryan [41], Hsu[43], Davies [19], and Takeda [64] for further discussions in terms of volume growth and bounds on Ricci curvature. For discussions on the behaviour at infinity of diffusion processes, and the C_0 property, we refer the reader to Azencott [4] and Elworthy [32]. But we would like to mention that a flow consisting of diffeomorphisms (c.f. page 79) has the C_0 property by arguing by contradiction as in [32].

Those papers above are on a Riemannian manifold except for the last reference. For a manifold without a Riemannian structure, Elworthy [31] following Itô [46] showed that the diffusion solution to (1.1) does not explode if there is a uniform cover for the coefficients of the equation. See also Clark[15]. In particular this shows that the s.d.e (1.1) does not explode on a compact manifold if the coefficients are reasonably smooth. See [10], [25]. To apply this method to check whether a Riemannian manifold is stochastically complete, we usually construct a stochastic differential equation whose solution is Brownian motion.

II. Main results

The main aim [53] of this chapter is to give unified treatment to some of the results from H. Donnelly-P. Li and L. Schwartz. It gives the first a probabilistic interpretation and extends part of the latter. We first introduce weak uniform covers in an analogous way to uniform covers, which gives non-explosion test by using estimations on exit times of the diffusion considered. As a corollary this gives the known result on nonexplosion of a Brownian motion on a complete Riemannian manifold with Ricci curvature going to negative infinity at most quadratically in the distance function[44].

One interesting example is that a solution to a stochastic differential equation on R^n whose coefficients have linear growth has no explosion and has the C_0 property. Notice under this condition, its associated generator has quadratic growth. On the other hand let $M = R^n$, and let L be an elliptic differential operator :

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i},$$

where a_{ij} and b_i are C^2. Let (s_{ij}) be the positive square root of the matrix (a_{ij}). Let $X^i = \sum_j s_{ij} \frac{\partial}{\partial x_i}, A = \sum_j b_j \frac{\partial}{\partial x_j}$. Then the s.d.e. defined by:

$$(It\hat{o}) \qquad dx_t = \sum_i X^i(x_t) dB_t^i + A(x_t) dt$$

has generator L. Furthermore if $|(a_{ij})|$ has quadratic growth and b_i has linear growth, then both X and A in the s.d.e. above have linear growth. In this case any solution u_t to the following partial differential equation :

$$\frac{\partial u_t}{\partial t} = L u_t$$

satisfies : $u_t \in C_0(M)$, if $u_0 \in C_0(M)$ (see next part).

III. Heat equations, semigroups, and flows

Let $\bar{M}$ be a compactification of M, i.e. a compact Hausdorff space which contains M as a dense subset. We assume $\bar{M}$ is first countable. Let h be a continuous function on $\bar{M}$. Consider the following heat equation with initial boundary conditions:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f, \quad x \in M, t > 0 \tag{4.1}$$

$$f(x,0) = h(x), x \in M \tag{4.2}$$

$$f(x,t) = h(x), x \in \partial M. \tag{4.3}$$

It is known that there is a unique minimal solution to the first two equations on a stochastically complete manifold, the solution is in fact given by the semigroup associated with Brownian motion on the manifold. So the above equation is not solvable in general. However with a condition imposed on the boundary of the compactification, Donnelly-Li [21] showed that the heat semigroup satisfies (4.3). Here is the condition and the theorem:

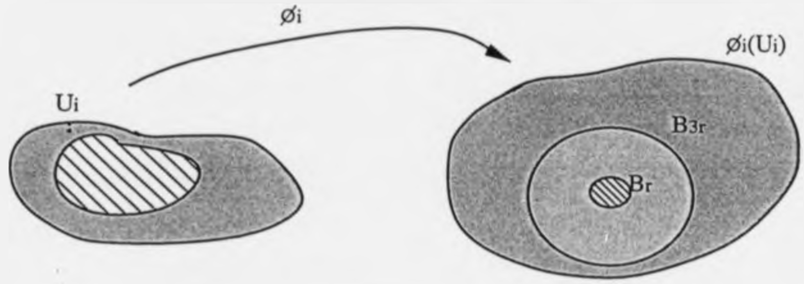
The ball convergence criterio n: Let $\{x_n\}$ be a sequence in M converging to a point x on the boundary, then the geodesic balls $B_r(x_n)$, centered at x_n radius r, converge to x as n goes to infinity for each fixed r.

An example of manifolds which satisfies the ball convergence criterion is R^n with sphere at infinity. However this is not true if R^n is given the compactification of a cylinder with a circle at infinity added at each end. The one point compactification also satisfies the ball convergence criterion. Another class of examples is manifolds with their geometric compactifications and with the cone topology, i.e. the boundary of the manifold are equivalent classes of geodesic rays. See [23]. Recall that two geodesic rays $\{\gamma_1(t), t \geq 0\}$ and $\{\gamma_2(t), t \geq 0\}$ are said to be equivalent if the the Riemannian distance $d(\gamma_1(t), \gamma_2(t))$ between the two points $\gamma_1(t)$ and $\gamma_2(t)$ is smaller than a constant for each t.

Theorem 4.1.1 (H.Donnelly-P. Li) *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. The over determined equation (4.1)-(4.3) is solvable for any given continuous function h on $\bar{M}$, if and only if the ball convergence criterion holds for M.*

Notice that if the Brownian motion starting from x converges to the same point on the boundary to which x converges, then (4.1)-(4.3) is clearly solvable. See section 4 for details. We would also like to consider the opposite question: Do we get any information on the diffusion processes if we know the behaviour at infinity of the associated semigroups? This is true for many cases. In particular, for the one point compactification, Schwartz has the following theorem[60], which provides a partial converse to [32]:

Theorem 4.1.2 (L. Schwartz) *Let $\bar{F}_t$ be the standard extension of F_t to $\bar{M} = M \cup \{\infty\}$, the one point compactification. Then the map $(t, x) \mapsto \bar{F}_t(x)$ is continuous from $R_+ \times \bar{M}$ to $L^0(\Omega, P, \bar{M})$, the space of measurable maps with topology induced from convergence in probability, if and only if the Semigroup P_t has the C_0 property and the map $t \mapsto P_t f$ is continuous from R_+ to $C_0(M)$.*



4.2 Weak uniform covers and nonexplosion

Definition 4.2.1 *[31] A stochastic dynamical system (1) is said to admit a uniform cover(radius $r > 0$, bound k), if there are charts $\{\phi_i, U_i\}$ of diffeomorphisms from open sets U_i of the manifold onto open sets $\phi_i(U_i)$ of R^n, such that:*

1. *$B_{3r} \subset \phi_i(U_i)$, each i. (B_α denotes the open ball about 0, radius α).*
2. *The open sets $\{\phi_i^{-1}(B_r)\}$ cover the manifold.*
3. *If $(\phi_i)_*(X)$ is given by:*

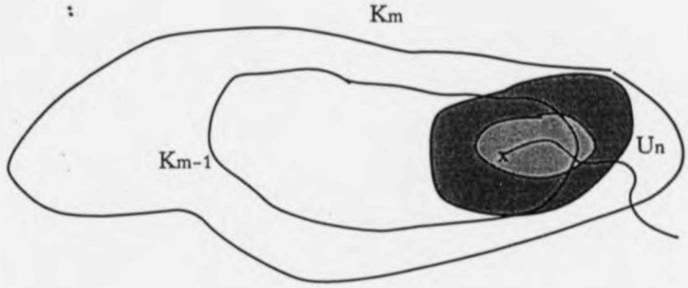
$$(\phi_i)_*(X)(v)(e) = (D\phi_i)_{\phi_i^{-1}(v)} X(\phi_i^{-1}v)(e)$$

with $(\phi_i)_*(A)$ similarly defined, then both $(\phi_i)_*(X)$ and $\mathcal{A}(\phi_i)$ are bounded by k on B_{2r}. Here $\mathcal{A}$ is the generator for the dynamical system.

Let $M = R^n$. Equation (1.1) can be interpreted as Itô integral.

Definition 4.2.2 *A diffusion process $\{F_t, \xi\}$ is said to have a weak uniform cover if there are pairs of connected open sets $\{U_n^0, U_n\}_{n=1}^{\infty}$, and a non-increasing sequence $\{\delta_n\}$ with $\delta_n > 0$, such that:*

1. *$U_n^0 \subset U_n$, and the open sets $\{U_n^0\}$ cover the manifold. For $x \in U_n^0$ denote by $\tau^n(x)$ the first exit time of $F_t(x)$ from the open set U_n. Assume $\tau^n < \xi$ a.s. unless $\tau^n = \infty$ almost surely.*



2. *There exists $\{K_n\}_{n=1}^{\infty}$, a family of increasing open subsets of M with $\cup K_n = M$, such that each U_n is contained in one of these sets and intersects at most one boundary from $\{\partial K_m\}_{m=1}^{\infty}$.*

3. *Let $x \in U_n^0$ and $U_n \subset K_m$, then for $t < \delta_m$:*

$$P\{\omega: \tau^n(x) < t\} \leq Ct^2. \tag{4.4}$$

4. $\sum_{n=1}^{\infty} \delta_n = \infty$.

Notice the introduction of $\{K_n\}$ is only for giving an order to the open sets $\{U_n\}$. This is quite natural when looking at concrete manifolds. In a sense the condition says the geometry of the manifold under consideration changes slowly as far as the diffusion process is concerned. In particular if the δ_n can be taken all equal, we take $K_n = M$; e.g. when the number of open sets in the cover is finite. On a Riemannian manifold the open sets are often taken as geodesic balls.

Lemma 4.2.1 *Assume there is a uniform cover for the stochastic dynamic system(1), then the solution has a weak uniform cover with $\delta_n = 1$, all n.*

Proof: This comes directly from lemma 5 , Page 127 in [31]. ∎

An example of a stochastic differential equation which has a weak uniform cover but not a uniform cover is given by example 0 on page 50 and the example on page 57.

Remarks:

1. If T is a stopping time, then the inequality (4.4) gives the following from the strong Markov property of the process.:

   let $V \subset U_n^0$, and $V \subset K_m$, then when $t < \delta_m$:

   $$P\{\tau^n(F_T(x)) < t | F_T(x) \in V\} \le Ct^2, \tag{4.5}$$

   since

   $$\begin{aligned} P & \{\tau^n(F_T(x)) < t, F_T(x) \in V\} \\ = & \int_V P(\tau^n(y) < t) P_T(x, dy) \le Ct^2 P\{F_T(x) \in V\}, \end{aligned}$$

   here $P_T(x, dy)$ denotes the distribution of $F_T(x)$.

2. Denote by $P_t^{U_n}$ the heat solution on U_n with Dirichlet boundary condition, then (4.4) is equivalent to the following: when $x \in U_n^0$,

   $$1 - P_t^{U_n}(1)(x) \le Ct^2. \tag{4.6}$$

3. The methods in the article work in infinite dimensions to give analogous results (when a Riemannian metric in not needed).

Exit times: Given such a cover, let $x \in U_n^0$. We define stopping times $\{T_k(x)\}$ as follows: Let $T_0 = 0$. Let $T_1(x) = \inf_t\{F_t(x,\omega) \notin U_n\}$ be the first exit time of $F_t(x)$ from the set U_n. Then $F_{T_1}(x,\omega)$ must be in one of the open sets $\{U_k^0\}$. Let

$$\begin{aligned} \Omega_1^1 &= \{\omega : F_{T_1}(x)(\omega) \in U_1^0, T_1(x,\omega) < \infty\}, \\ \Omega_k^1 &= \{\omega : F_{T_1}(x) \in U_k^0 - \bigcup_{j=1}^{k-1} U_j^0, T_1 < \infty\}. \end{aligned}$$

Then $\{\Omega_k^1\}$ are disjoint sets such that $\cup\Omega_k^1 = \{T_1 < \infty\}$. In general we only need to consider the nonempty sets of such. Define further the following: Let $T_2 = \infty$, if $T_1 = \infty$. Otherwise if $\omega \in \Omega_k^1$, let:

$$T_2(x,\omega) = T_1(x,\omega) + \tau^k(F_{T_1}(x,\omega)). \tag{4.7}$$

In a similar way, the whole sequence of stopping times $\{T_j(x)\}$ and sets $\{\Omega_k^j\}_{k=1}^{\infty}$ are defined for $j = 3, 4, \ldots$. Clearly Ω_k^j so defined is measurable with respect to the sub-algebra $\mathcal{F}_j$.

Lemma 4.2.2 *Given a weak uniform cover as above. Let $x \in U_n^0$ and $U_n \subset K_m$. Let $t < \delta_{m+k}$. Then*

$$P\{\omega : T_k(x,\omega) - T_{k-1}(x,\omega) < t, T_{k-1} < \infty\} \leq Ct^2. \tag{4.8}$$

Proof: Notice for such x, $F_{T_k}(x) \in K_{m+k-1}$. Therefore for $t < \delta_{m+k}$ we have:

$$\begin{aligned}
P \quad & \{\omega : T_k(x,\omega) - T_{k-1}(x,\omega) < t, T_{k-1} < \infty\} \\
= \quad & \sum_{j=1}^{\infty} P(\{\omega : T_k(x,\omega) - T_{k-1}(x,\omega) < t\} \cap \Omega_j^{k-1}) \\
= \quad & \sum_{j=1}^{\infty} P\{\tau^j(F_{T_{k-1}}(x)) < t, \Omega_j^{k-1}\} \\
\leq \quad & Ct^2 \sum_{j=1}^{\infty} P(\Omega_j^{k-1}) \leq Ct^2,
\end{aligned}$$

as in remark 1. Here χ_A is the characteristic function for a measurable set A, and E denotes taking expectation. ∎

Lemma 4.2.3 *If $\sum_n t_n = \infty$, $t_n > 0$ non-increasing. Then there is a non-increasing sequence $\{s_n\}$, such that $0 < s_n \leq t_n$:*

(i) $\sum s_n = \infty$

(ii) $\sum s_n^2 < \infty$

Proof: Assume $t_n \leq 1$, all n. Group the sequence $\{t_n\}$ in the following way:

$$t_1;\ t_2,\ \ldots,\ t_{k_2};\ t_{k_2+1},\ \ldots,\ t_{k_3};\ t_{k_3+1}\ \ldots$$

such that $1 \leq t_2 + \ldots + t_{k_2} \leq 2$, $1 \leq t_{k_i+1} + \ldots + t_{k_{i+1}} \leq 2$, $i \geq 2$. Let $s_1 = t_1$, $s_2 = \frac{t_2}{2}, \ldots, s_{k_2} = \frac{t_{k_2}}{2}$, $s_{k_2+1} = \frac{t_{k_2+1}}{3}, \ldots, s_{k_3} = \frac{t_{k_3}}{3}$, $s_{k_3+1} = \frac{t_{k_3+1}}{4}, \ldots$. Clearly the s_n's so defined satisfy the requirements. ■

So without losing generality, we may assume from now on that the constants $\{\delta_n\}$ for a weak uniform cover fulfill the two conditions in the above lemma. With these established, we can now state the nonexplosion result. The proof is analogous to that of theorem 6 on Page 129 in [31].

Theorem 4.2.4 *If the solution $F_t(x)$ of the equation* (1) *has a weak uniform cover, then it is complete(nonexplosion).*

Proof: Let $x \in K_n$, $t > 0$, $0 < \epsilon < 1$. Pick up a number p (possibly depending on ϵ and n), such that $\sum_{i=n+1}^{n+p} \epsilon\delta_i > t$. This is possible since $\sum_{i=1}^{\infty} \delta_i = \infty$. So

$$\begin{aligned}
P\{\xi(x) < t\} &\leq P\{T_p(x) < t, T_{p-1} < \infty\} \\
&= P\{\sum_{k=1}^{p}(T_k(x) - T_{k-1}(x)) < t, T_{p-1} < \infty\} \\
&\leq \sum_{k=1}^{p} P\{T_k(x) - T_{k-1}(x) < \epsilon\delta_{n+k}, T_{k-1} < \infty\} \\
&\leq Ct^2\epsilon^2 \sum_{k=n}^{n+p} \delta_k^2 \leq Ct^2\epsilon^2 \sum_{k=1}^{\infty} \delta_k^2.
\end{aligned}$$

Let $\epsilon \to 0$, we get: $P\{\xi(x) < t\} = 0$. ■

Remark: The argument in the above proof is valid if the definition of a weak uniform cover is changed slightly, i.e. replacing the constant C by C_n (with some slow growth condition, say $\frac{1}{p^2}\sum_{j=1}^{p} C_{n+j}$ is bounded for all n) but keep all δ_n equal.

As a corollary, we have the following known result:

Corollary 4.2.5 *Let ξ be the explosion time. If $P\{\xi < t_0\} = 0$, for some $t_0 > 0$. Then there is no explosion.*

Prove by induction. In the following we look into some examples:

Example 0: The flow $F_t(x) = x + B_t$ on $R^n - \{0\}$ does not have a uniform cover. The problem occurs at the origin. But it does have a weak uniform cover as constructed below. First note that we only need to worry about the origin. Take $U_n = \{x : |x| \le a_n\}$, for $a_n = \left(\frac{1}{n+1}\right)^{\frac{1}{8}}$. Let $U_n^0 = U_{n-1}$, $K_n = U_n$, and $C_n = k\sqrt{n}$. Here k is a constant. Now

$$\begin{aligned} P\{\tau_n < t\} &\le P\{\sup_{s\le t} |B_s| \le a_n - a_{n+1}\} \\ &\le \frac{kt^2}{[a_n - a_{n+1}]^4} \le C_n t^2 \end{aligned}$$

by the maximal inequality. Then

$$\lim_{p\to\infty} \frac{1}{p^2} \sum_1^p C_{n+j} = 0.$$

See the remark above to see $\{U_n\}$ is a "weak uniform cover".

Example 1 : Let $\{U_n\}$ be a family of relatively compact open (proper) subsets of M such that $U_n \subset U_{n+1}$ and $\cup_{n=1}^{\infty} U_n = M$. Assume there is a sequence of numbers $\{\delta_n\}$ with $\sum_n \delta_n = \infty$, such that the following inequality holds when $t < \delta_{n-1}$ and $x \in U_{n-1}$: $P\{\tau^{U_n}(x) < t\} \le ct^2$. Then the diffusion concerned does not explode by taking $\{U_{n+1} - \overline{U_{n-1}}, U_n - \overline{U_{n-1}}\}$ to be a weak uniform cover and $K_n = U_n$.

4.3 Boundary behaviour of diffusion processes

To consider the boundary behaviour of diffusion processes, we introduce the following concept:

Definition 4.3.1 *A weak uniform cover $\{U_n^0, U_n\}$ is said to be regular (at infinity for M), if the following holds: let $\{x_j\}$ be a sequence in M converging to $x \in \partial M$, and $x_j \in U_{n_j}^0 \in \{U_n^0\}_{n=1}^\infty$, then the corresponding open sets $\{U_{n_j}\}_{j=1}^\infty \subset \{U_n\}_1^\infty$ converges to x as well. A regular uniform cover can be defined in a similar way.*

For a point x in M, there are a succession of related open sets $\{W_x^p\}_{p=1}^\infty$, which are defined as follows: Let W_x^1 be the union of all open sets from $\{U_n\}$ such that U_n^0 contains x, and W_k^2 be the union of all open sets from $\{U_n\}$ such that U_n^0 intersects one of the small balls $U_{n_j}^0$ defining W_x^1. Similarly $\{W_k^p\}$ are defined. These sets are well defined and in fact form an increasing sequence.

Lemma 4.3.1 *Assume F_t has a regular (weak) uniform cover. Let $\{x_n\}$ be a sequence of M which converges to a point $x \in \partial M$. Then $W_{x_n}^p$ converges to x as well for each fixed p.*

Proof: Note that by arguing by contradiction, we only need to prove the following: let $\{z_k\}$ be a sequence $z_k \in W_{x_n}^p$, then $z_k \to x$, as $n \to \infty$. First let $p = 2$.

By definition, for each x_k, z_k, there are open sets $U_{n_k}^0$ and $U_{m_k}^0$ such that $x_k \in U_{n_k}^0$, $z_k \in U_{m_k}^0$ and $U_{n_k}^0 \cap U_{m_k}^0 \neq \emptyset$. Furthermore $U_{n_k} \to x$ as $k \to \infty$. Let $\{y_k\}$ be a sequence of points with $y_k \in U_{n_k}^0 \cap U_{m_k}^0$. But $y_k \to x$ since U_{n_k} does. So $U_{m_k} \to x$ again from the definition of a regular weak uniform cover. Therefore z_k converges to x as $k \to \infty$, which is what we want. The rest can be proved by induction. ∎

Theorem 4.3.2 *If the diffusion F_t admits a regular weak uniform cover, with $\delta_n = \delta$, all n, then the map $F_t(-) : M \to M$ can be extended to the compactification M continuously in probability with the restriction to the boundary to be the identity map, uniformly in t in finite intervals. (We will say F_t extends.)*

Proof: Take $x \in \partial M$ and a sequence $\{x_n\}$ in M converging to x. Let U be a neighbourhood of x in ∂M. We want to prove for each t:

$$\lim_{n\to\infty} P\{\omega : F_s(x_n,\omega) \notin U, s < t\} = 0.$$

Since x_n converges to x, there is a number $N(p)$ for each p, such that if $n > N$, $W^p_{x_n} \subset U$. Let $t > 0$, Choose p such that $\frac{2t}{p} < \delta$. For such a number $n > N(p)$ fixed, we have:

$$\begin{aligned}
P & \quad \{\omega : F_s(x_n,\omega) \notin U, s < t\} \\
\leq & \quad P\{\omega : F_s(x,\omega) \notin W^p_{x_n}, s < t\} \\
\leq & \quad P\{\omega : T_p(x_n)(\omega) < t, T_{p-1}(x_n) < \infty\} \\
\leq & \quad \sum_{k=1}^{p} P\{T_k(x_n) - T_{k-1}(x_n) < \frac{t}{p}, T_{k-1}(x_n) < \infty\}. \\
\leq & \quad \frac{Ct^2}{p}.
\end{aligned}$$

Here C is the constant in the definition of the weak uniform cover. Let p go to infinity to complete the proof. ■

Remark: If the δ_n can be taken all equal, theorem 4.2.4 , theorem 4.3.2 hold if (4.4) is relaxed to:

$$P\{T^n(x) < t\} \leq f(t),$$

for some nonnegative function f satisfying $\lim_{t\to 0} \frac{f(t)}{t} = 0$.

Example 2: Let $M = R^n$ with the one point compactification. Consider the following s.d.e.:

$$(It\hat{o}) \quad dx_t = X(x_t)dB_t + A(x_t)dt.$$

Then if both X and A have linear growth, the solution has the C_0 property.

Proof: There is a well known uniform cover for this system. See [15], or [31]. A slight change gives us the following regular uniform cover:

Take a countable set of points $\{p_n\}_{n\geq 0} \subset M$ such that the open sets $U_n^0 = \{z : |z - p_n| < \frac{|p_n|}{3}\}$, $n = 1, 2, \ldots$ and $U_0^0 = \{z : |z - p_0| < 2\}$ cover R^n. Let $U_0 = \{z : |z - p_0| < 6\}$; and $U_n = \{z : |z - p_n| < \frac{|p_n|}{2}\}$, for $n \neq 0$. Let ϕ_n be the chart map on U_n:

$$\phi_n(z) = \frac{z - p_n}{|p_n|}.$$

This certainly defines a uniform cover(for details see Example 3 below). Furthermore if $z_n \to \infty$ and $z_n \in U_n^0$, then any $y \in U_n$ satisfies the following:

$$|y| > \frac{|p_n|}{2} > \frac{1}{3}|z_n| \to \infty,$$

since $|p_n| \geq \frac{3|z_n|}{4}$. Thus we have a regular uniform cover which gives the required C_0 property. ■

Example 3: Let $M = R^n$, compactified with the sphere at infinity: $\bar{M} = R^n \cup S^{n-1}$. Consider the same s.d.e as in the example above. Suppose both X and A have sublinear growth of power $\alpha < 1$:

$$\begin{aligned} |X(x)| &\leq K(|x|^\alpha + 1) \\ |A(x)| &\leq K(|x|^\alpha + 1) \end{aligned}$$

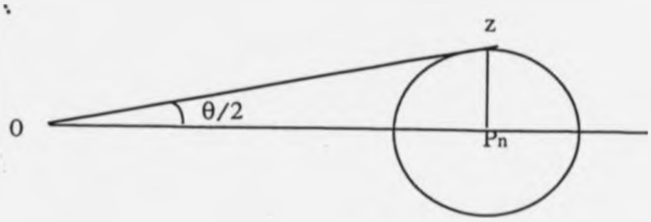
for a constant K. Then there is no explosion. Moreover the solution F_t extends.

Proof: The proof is as in example 2, we only need to construct a regular uniform cover for the s.d.e.:

Take points $p_0, p_1, p_2, \ldots$ in R^n (with $|p_0| = 1$, $|p_n| > 1$) , such that the open sets $\{U_n^0\}$ defined by : $U_0 = \{z : |z - p_0| < 2\}$, $U_i^0 = \{z : |z - p_i| < \frac{|p_i|^\alpha}{6}\}$ cover R^n.

Let $U_0 = \{z : |z - p_0| < 6\}$, $U_i = \{z : |z - p_i| < \frac{|p_i|^\alpha}{2}\}$, and let ϕ_i be the chart map from U_i to R^n:

$$\phi_i(z) = \frac{6(z - p_i)}{|p_i|^\alpha}.$$



Then $\{\phi_i, U_i\}$ is a uniform cover for the stochastic dynamical system. In fact, for $i \neq 0$, and $y \in B_3 \subset R^n$:

$$|(\phi_i)_*(X)(y)| \leq \frac{K(1+|\phi_i^{-1}(y)|^\alpha)}{|p_i|^\alpha} \leq \frac{K}{|p_i|^\alpha}(1+2|p_i|^\alpha) < 18K.$$

Similarly $|(\phi_i)_*(A)(y)| \leq 18K$, and $D^2\phi_i = 0$.

Next we show this cover is regular. Take a sequence x_k converging to $\bar{x}$ in ∂R^n. Assume $x_k \in U_k^0$. Let $z_k \in U_k$. We want to prove $\{z_k\}$ converge to $\bar{x}$. First the norm of z_k converges to infinity as $k \to \infty$, since $|p_k| > \frac{2|x_k|}{3}$ and $|z_k| > |p_k| - \frac{1}{2}|p_k|^\alpha$.

Let θ be the biggest angle between points in U_k, then

$$\tan\frac{\theta}{2} \leq \sup_{z \in U_n} \frac{|z - p_n|}{|p_n|} \leq \frac{|p_n|^\alpha}{2|p_n|} \leq \frac{|p_n|^{\alpha-1}}{2} \to 0.$$

Thus $\{U_n, \phi_n\}$ is a uniform cover satisfying the convergence criterion for the sphere compactification. The required result holds from the theorem. ■

This result is sharp in the sense there is a s.d.e. with coefficients having linear growth but the solution to it does not extend to the sphere at infinity to be identity:

Example 4: Let B be a one dimensional Brownian motion. Consider the following s.d.e on the complex plane $\mathcal{C}$:

$$dx_t = ix_t dB_t.$$

The solution starting from x is in fact $xe^{iB_t+\frac{t}{2}}$, which does not continuously extend to be the identity on the sphere at infinity.

Example 5: Let U be a bounded open set of R^n. Let (X, A) be a s.d.s. (in Itô form) on U satisfying

$$|X(x)| \leq k\ d(x, \partial U),$$

and

$$|A(x)| \leq k\ d(x, \partial U)$$

for some constant k, then there is no explosion. Here d denotes the distance function on R^n and ∂U denotes the boundary of U.

Proof: Choose points $\{x_n\}$ such that the balls $B(x_n, \frac{1}{2}d(x_n, \partial U))$ centered at x_n radius $\frac{1}{2}d(x_n, \partial U)$ cover U. Define a map:

$$\phi_n : B(x_n, \frac{1}{2}d(x_n, \partial U)) \to B_3$$

by

$$\phi_n(x) = 6 \times \frac{x - x_n}{d(x_n, \partial U)}.$$

Then

$$(T\phi_n)_*(X) \leq \frac{6|X(\phi_n^{-1}(x))|}{d(x_n, \partial U)} \leq \frac{6kd(\phi_n^{-1}(x), \partial U)}{d(x_n, \partial U)} \leq 12k$$

by the triangle inequality of the distance function. Thus we have a uniform cover and so nonexplosion. ■

This result is sharp in the sense that there is an example (in Itô form) given in [65] which satisfies:

$$\frac{|X(x)| + |A(x)|}{d(x, \partial U)^\epsilon} \leq 2$$

for $0 < \epsilon < 1$, but has explosion. Here is the example (on $\{|x| < 1\} \subset R^2$):

$$\begin{aligned} dx_t^1 &= (1 - |x_t|^2)^\epsilon x_t^1 dB_t - \frac{1}{2}(1 - |x_t|^2)^{2\epsilon} x_t^1\, dt \\ dx_t^2 &= (1 - |x_t|^2)^\epsilon x_t^2 dB_t - \frac{1}{2}(1 - |x_t|^2)^{2\epsilon} x_t^2\, dt. \end{aligned}$$

Here $0 < \epsilon < 1$. See [65] for more discussions on nonexplosion on open sets of R^n.

4.4 Boundary behaviour continued

A diffusion process is a C_0 diffusion if its semigroup has the C_0 property. This is equivalent to the following [4]: let K be a compact set, and $T_K(x)$ the first entrance time to K of the diffusion starting from x, then $\lim_{x\to\infty} P\{T_K(x) < t\} = 0$ for each $t > 0$, and each compact set K.

The following theorem follows from theorem 3.2 when δ_n in the definition of weak uniform cover can be taken all equal:

Theorem 4.4.1 *Let M be the one point compactification. Then if the diffusion process $F_t(x)$ admits a regular weak uniform cover, it is a C_0 diffusion.*

Proof: Let K be a compact set with $K \subset K_j$; here $\{K_j\}$ is as in definition 2.2. Let $\epsilon > 0$, $t > 0$, then there is a number $N = N(\epsilon, t)$ such that:

$$\delta_{j+2} + \delta_{j+4} + \ldots + \delta_{j+2N-2} > \frac{t}{\epsilon}.$$

Take $x \notin K_{j+2N}$. Assume $x \in K_m$, some $m > j + 2N$. Let T_0 be the first entrance time of $F_t(x)$ to K_{j+2N-1}, T_1 be the first entrance time of $F_t(x)$ to K_{j+2N-3} after T_0, (if $T_0 < \infty$), and so on. But $P\{T_i < t, T_{i-1} < \infty\} \leq Ct^2$ for $t < \delta_{j+2N-2i}$, $i > 0$, since any open sets from the cover intersects at most one boundary of sets from $\{K_n\}$. Thus

$$\begin{aligned} P\{T_K(x) < t\} &\le P\{\sum_1^{N-1} T_i(x) < t, T_{N-2} < \infty\} \\ &\le \sum_{i=1}^{N-1} P\{T_i(x) < \epsilon\delta_{j+2N-2i}, T_{i-1}(x) < \infty\} \\ &\le C\epsilon^2 \sum_1^{N-1} \delta^2_{j+2N-2i} \le C\epsilon^2 \sum_1^{\infty} \delta_j^2. \end{aligned}$$

The proof is complete by letting $\epsilon \to 0$. ■

Example 6: Let M be a complete Riemannian manifold, p a fixed point in M. Denote by $\rho(x)$ the distance between x and p, $B_r(x)$ the geodesic ball centered at x radius r, and Ricci(x) the Ricci curvature at x.

Assumption A:

$$\int_1^\infty \frac{1}{\sqrt{K(r)}} dr = \infty. \tag{4.9}$$

Here K is defined as follows:

$$K(r) = -\{\inf_{B_r(p)} \text{Ricci}(x) \wedge 0\}.$$

Let $X_t(x)$ be a Brownian motion on M with $X_0(x) = x$. Consider the first exit time of $X_t(x)$ from $B_1(x)$:

$$T = \inf_t\{t \ge 0 : \rho(x, X_t(x)) = 1\}.$$

Then we have the following estimate on T from [43]:

If $L(x) > \sqrt{K(\rho(x)+1)}$, then

$$P\{T(x) \le \frac{c_1}{L(x)}\} \le e^{-c_2 L(x)}$$

for all $x \in M$. Here c_1, c_2 are positive constants independent of L.

This can be rephrased into the form we are familiar with: when $0 \leq t < \frac{c_1}{\sqrt{K(\rho(x)+1)}}$,

$$P\{T(x) \leq t\} \leq e^{-\frac{c_1 c_2}{t}}.$$

But $\lim_{t \to 0} \frac{e^{-\frac{c_1 c_2}{t}}}{t^2} = 0$. So there is a $\delta_0 > 0$, such that: $e^{-\frac{c_1 c_2}{t}} \leq t^2$, when $t < \delta_0$. Thus:

Estimation on exit times: when $t < \frac{c_1}{\sqrt{K(\rho(x)+1)}} \wedge \delta_0$,

$$P\{T(x) < t\} \leq t^2. \tag{4.10}$$

Let $\delta_n = \frac{1}{\sqrt{K(3n+1)}} \wedge \delta_0$, then we also have the following :

$$\sum_1^\infty \delta_n \geq \sum_1^\infty \frac{1}{\sqrt{K(3n+1)}} \geq \int_1^\infty \frac{1}{\sqrt{K(3r)}}\, dr = \infty. \tag{4.11}$$

With this we may proceed to prove the following from [43]:

Corollary: [Hsu] A complete Riemannian manifold M with Ricci curvature satisfies assumption A is stochastically complete and has the C_0 property.

Proof: There is a regular weak uniform cover as follows:

First take any $p \in M$, and let $K_n = \overline{B_{3n}(p)}$. Take points p_i such that $U_i^0 = B_1(p_i)$ covers the manifold. Let $U_i = B_2(p_i)$. Then $\{U_i^0, U_i\}$ is a regular weak uniform cover for $M \cup \Delta$.

Remark: Grigoryan has the following volume growth test on nonexplosion. The Brownian motion does not explode on a manifold if

$$\int^\infty \frac{r}{\mathrm{Ln}(\mathrm{Vol}(B_R))}\, dr = \infty.$$

Here $\mathrm{Vol}(B_R)$ denotes the volume of a geodesic ball centered at a point p in M. This result is stronger than the corollary obtained above by the following comparison theorem on a n dimensional manifold: let ω_{n-1} denote the volume of the $n-1$ sphere of radius 1,

$$\mathrm{Vol}(B_R) \le \omega_{n-1} \int_0^R \{\sqrt{\frac{(n-1)}{K(R)}}\mathrm{Sinh}(\sqrt{\frac{K(R)}{(n-1)}}r)\}^{(n-1)}\, dr.$$

Notice $K(R)$ is positive when R is sufficiently big provided the Ricci curvature is not nonnegative everywhere. So Grigoryan's result is stronger than the one obtained above.

The definition of weak uniform cover is especially suitable for the one point compactification. For general compactification the following definition explores more of the geometry of the manifold and gives better result:

Definition 4.4.1 *Let $\bar{M}$ be a compactification of M, $x \in \partial M$. A diffusion process F_t is said to have a uniform cover at point x, if there is a sequence A_n of open neighbourhoods of x in $\bar{M}$ and positive numbers δ_n and a constant $c > 0$, such that:*

1. *The sequence of A_n is strictly decreasing, with $\cap A_n = x$, and $A_n \supset \partial A_{n+1}$.*

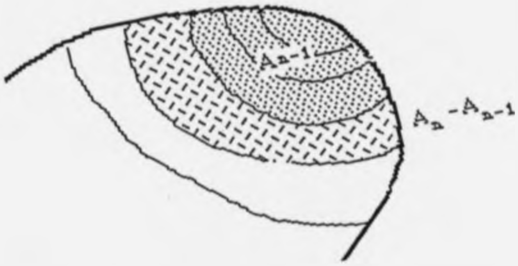
2. *The sequence of numbers δ_n is non-increasing with $\sum \delta_n = \infty$ and $\sum_n \delta_n^2 < \infty$.*

3. *When $t < \delta_n$, and $x \in A_n - A_{n+1}$,*

$$P\{\tau^{A_{n-1}}(x) < t\} \le ct^2.$$

Here $\tau^{A_n}(x)$ denotes the first exit time of $F_t(x)$ from the set A_n.

Proposition 4.4.2 *If there is a uniform cover for $x \in \partial M$, then $F_t(x)$ converges to x continuously in probability as $x \to x$ uniformly in t in finite interval.*



Proof: The existence of $\{A_n\}$ will ensure $F_{\tau^{A_n}}(x) \subset A_{n-1}$, which allows us to apply a similar argument as in the case of the one point compactification. Here we denote by τ^A the first exit time of the process $F_t(x)$ from a set A.

Let U be a neighbourhood of $\bar{x}$. For this U, by compactness of $\bar{M}$, there is a number m such that $A_m \subset U$, since $\cap_{k=1}^{\infty} A_k = \bar{x}$. Let $0 < \epsilon < 1$, $\bar{\epsilon} = (\frac{\epsilon}{c\sum_k \delta_k^2})^{\frac{1}{2}}$. we may assume $\bar{\epsilon} < 1$. Choose $p = p(\epsilon) > 0$ such that:

$$\delta_m + \delta_{m+1} + \ldots + \delta_{m+p-1} > \frac{t}{\bar{\epsilon}}.$$

Let $x \in A_{m+p+2}$. Denote by $T_0(x)$ the first exit time of $F_t(x)$ from A_{m+p+1}, $T_1(x)$ the first exit time of $F_{T_0}(x)$ from A_{m+p} where defined. Similarly $T_i, i > 1$ are defined.

Notice if $T_i(x) < \infty$, then $F_{T_i(x)} \in A_{m+p-i} - A_{m+p+1-i}$, for $i = 0, 1, 2, \ldots$. Thus for $i > 0$ there is the following inequality from the definition and the Markov property:

$$P\{T_i(x) < \bar{\epsilon}\delta_{m+p-i}\} \leq c\bar{\epsilon}^2\delta_{m+p-i}^2.$$

Therefore we have:

$$P\{\tau^U(x) < t\} \quad \leq \quad P\{\tau^{A_m}(x) < t\}$$

$$
\begin{aligned}
&\leq P\{T_p + \ldots + T_1 < t, T_{p-1} < \infty\} \\
&\leq \sum_{i=1}^{p} P\{T_i < \epsilon\delta_{m+p-i}, T_{i-1} < \infty\} \\
&\leq c\bar{\epsilon}^2 \sum_{i=1}^{p} \delta^2_{m+p-i} < \epsilon.
\end{aligned}
$$

The proof is finished. ■

4.5 Properties at infinity of semigroups

Recall a semigroup is said to have the C_0 property, if it sends $C_0(M)$, the space of continuous functions on M vanishing at infinity, to itself. Let $\bar{M}$ be a compactification of M. Denote by Δ the point at infinity for the one point compactification. Corresponding to the C_0 property of semigroups we consider the following C_* property for $\bar{M}$:

Definition 4.5.1 *A semigroup P_t is said to have the C_* property for $\bar{M}$, if for each continuous function f on $\bar{M}$, the following holds: let $\{x_n\}$ be a sequence converging to x in ∂M, then*

$$\lim_{n\to\infty} P_t f(x_n) = f(x), \tag{4.12}$$

To justify the definition, we notice if $\bar{M}$ is the one point compactification, condition C_* will imply the C_0 property of the semigroup. On the other hand if P_t has the C_0 property, it has the C_* property for $M \cup \Delta$ assuming nonexplosion. This is observed by subtracting a constant function from a continuous function f on $M \cup \Delta$: Let $g(x) = f(x) - f(\Delta)$, then $g \in C_0(M)$. So $P_t g(x) = P_t f(x) - f(\Delta)$. Thus

$$\lim_{n\to\infty} P_t f(x_n) = \lim_{n\to\infty} P_t g(x_n) + f(\Delta) = f(\Delta),$$

if $\lim_{n\to\infty} x_n = \Delta$.

In fact the C_* property holds for the one point compactification if and only if there is no explosion and the C_0 property holds. These properties are often possessed by processes, e.g. a Brownian motion on a Riemannian manifold with Ricci curvature which satisfies (4.9) has this property.

Before proving this claim, we observe first that:

Lemma 4.5.1 *If P_t has the C_* property for any compactification M, it must have the C_* property for the one point compactification.*

Proof: Let $f \in C(M \cup \Delta)$. Define a map β from M to $M \cup \Delta$: $\beta(x) = x$ on the interior of M, and $\beta(x) = \Delta$, if x belongs to the boundary. Then β is a continuous map from M to $M \cup \Delta$, since for any compact set K, the inverse set $M - K = \beta^{-1}(M \cup \Delta - K)$ is open in M.

Let g be the composition map of f with β: $g = f \circ \beta : M \to R$. Thus $g(x)|M = f(x)|M$, and $g(x)|_{\partial M} = f(\Delta)$. So for a sequence $\{x_n\}$ converging to $x \in \partial M$, $\lim_n P_t f(x_n) = \lim_n P_t g(x_n) = g(x) = f(\Delta)$. ■

We are ready to prove the following theorem:

Theorem 4.5.2 *If a semigroup P_t has the C_* property, the associated diffusion process F_t is complete.*

Proof: We may assume the compactification under consideration is the one point compactification from the the lemma above. Take $f \equiv 1$, $P_t f(x) = P\{t < \xi(x)\}$. But $P\{t < \xi(x)\} \to 1$ as $x \to \Delta$ from the assumption. More precisely for any $\epsilon > 0$, there is a compact set K_ϵ such that if $x \notin K_\epsilon$, $P\{t < \xi(x)\} > 1 - \epsilon$.

Let K be a compact set containing K_ϵ. Denote by τ the first exit time of $F_t(x)$ from K. So $F_\tau(x) \notin K_\epsilon$ on the set $\{\tau < \infty\}$. Thus:

$$\begin{aligned} P\{t < \xi(x)\} &\geq P\{\tau < \infty, t < \xi(F_\tau(x))\} + P\{\tau = \infty\} \\ &= E\{\chi_{\tau<\infty} E\{\chi_{t<\xi(F_\tau(x))} | \mathcal{F}_\tau\}\} + P\{\tau = \infty\}. \end{aligned}$$

Here χ_A denotes the characteristic function of set A. Applying the strong Markov property of the diffusion we have:

$$E\{\chi_{\tau<\infty}E\{\chi_{t<\xi(F_\tau(x))}|\mathcal{F}_\tau\}\} = E\{\chi_{\tau<\infty}E\{t<\xi(y)|F_\tau = y\}\}.$$

However

$$E\{\chi_{t<\xi(y)}|F_\tau = y\} > 1-\epsilon,$$

So

$$\begin{aligned} P\{t<\xi(x)\} &\geq P\{\tau=\infty\} + E\{\chi_{\tau<\infty}(1-\epsilon)\} \\ &= 1-\epsilon P\{\tau<\infty\}. \end{aligned}$$

Therefore $P\{t<\xi(x)\}=1$, since ϵ is arbitrary. ■

In the following we examine the the relation between the behaviour at ∞ of a diffusion process and the diffusion semigroup.

Theorem 4.5.3 *The semigroup P_t has the C_* property if and only if the diffusion process F_t is complete and can be extended to $\bar{M}$ continuously in probability with $F_t(x)|_{\partial M} = x$.*

Proof: Assume F_t is complete and extends. Take a point $\bar{x}\in\bar{M}$, and sequence $\{x_n\}$ converging to $\bar{x}$. Thus

$$\lim_{n\to\infty} P_t f(x_n) = \lim_{n\to\infty} Ef(F_t(x_n)) = Ef(\bar{x}) = f(\bar{x})$$

for any continuous function on $\bar{M}$, by the dominated convergence theorem.

On the other hand P_t does not have the C_* property if the assumption above is not true. In fact let x_n be a sequence converging to $\bar{x}$, such that for some neighbourhood U of $\bar{x}$, and a number $\delta>0$:

$$\underline{\lim}_{n\to\infty} P\{F_t(x_n)\notin U\} = 1-\delta.$$

There is therefore a subsequence $\{x_{n_i}\}$ such that:

$$\lim_{i\to\infty} P\{F_t(x_{n_i})\notin U\} = 1-\delta.$$

Thus there exists $N > 0$, such that if $i > N$:

$$P\{F_t(x_{n_i}) \in M - U\} > 1 - \frac{\delta}{2}.$$

But since M is a compact Hausdorff space, there is a continuous function f from M to $[0, 1]$ such that $f|_{M-U} = 1$, and $f|_G = 0$, for any open set G in U. Therefore

$$\begin{aligned} P_t f(x_n) &= Ef(F_t(x_n)) \\ &\geq \int_{\{\omega: F_t(x_n) \in \bar{M} - U\}} f(F_t(x_n))\, P(d\omega) \\ &= P\{F_t(x_n) \in \bar{M} - U\} > \frac{\delta}{2}. \end{aligned}$$

So $\lim P_t f(x_n) \neq f(x) = 0$. ■

As is known, a flow consisting of diffeomorphisms has the C_0 property. But this is , in general, not true for the C_* property. See example 4.

Corollary 4.5.4 *Assume the diffusion process F_t admits a weak uniform cover regular for $M \cup \Delta$, then its diffusion semigroup p_t has the C_* property. The same is true for a general compactification if all δ_n in the weak uniform cover can be taken equal.*

Example 7:[21] Let M be a complete connected Riemannian manifold with Ricci curvature bounded from below. Let $\bar{M}$ be a compactification such that the ball convergence criterion holds (ref. section 1). In particular the over determined equation (4.1)-(4.3) is solvable for any continuous function f on $\bar{M}$ if the ball convergence criterion holds.

Proof: We keep the notation of example 6. Let $K = -\{\inf_x Ricci(x) \wedge 0\}$, $\delta = \frac{c_1}{k}$, where c_1 is the constant in example 6. Let $p \in M$ be a fixed point, and $K_n = \overline{B_{4n}(p)}$ be compact sets in M. Take points $\{p_i\}$ in M such that $\{B_1(p_i)\}$ cover the manifold. Then $\{B_1(p_i), B_2(p_i)\}$ is a weak uniform cover from (4.10) and (4.11). Moreover this is a regular cover if the ball convergence criterion holds for the compactification.

Part III

Strong completeness, derivative semigroup, and moment stability

Chapter 5

On the existence of flows: strong completeness

5.1 Introduction

A stochastic dynamical system (X, A) is said to be *complete* if its explosion time $\xi(x)$ is infinite almost surely for each x in M. It is called *strongly complete* if there is a version of the solution which is jointly continuous in time and space for all time. In this case the solution is called a *continuous flow*. Examples of s.d.s. which are complete but not strongly complete can be found in [31], [47], and [52].

The known results on the existence of a continuous flow are concentrated on R^n and compact manifolds. On R^n results are given (for Itô equations) in terms of global Lipschitz or similar conditions. See Blagovescenskii and Friedlin [9]. The problems concerning the diffeomorphism property of flows have been discussed by e.g. Kunita [51], Carverhill and Elworthy [10]. See Taniguchi [65] for discussions on the strong completeness of a stochastic dynamical system on an open set of R^n. For discussions of higher derivatives of solution flows on R^n, see Krylov [50] and Norris [58].

On a compact manifold, a stochastic differential equation with C^2 coefficients is strongly complete. In fact the solution flow is almost as smooth as the coefficients of the stochastic differential equation. More precisely the solution flow is C^{r-1} if the coefficients are C^r. Moreover the flow consists of diffeomorphisms. See Kunita [51], Elworthy [30], and Carverhill and Elworthy [10]. For discussions in the framework of diffeomorphism groups see Baxendale [6] and Elworthy [31].

In general we know very little about strong completeness. Our aim is to prove strong completeness given nonexplosion and certain regularity properties of the solution.

To begin with we quote the following theorem on the existence of a *partial flow* from [31] (first proved in [51], extended later in [31], and [10]):

Theorem 5.1.1 *[31] Suppose* X, *and* A *are* C^r, *for* $r \geq 2$. *Then there is a partially defined flow* $(F_t(\cdot), \xi(\cdot))$ *which is a maximal solution to* (1.1) *such that if*

$$M_t(\omega) = \{x \in M, t < \xi(x,\omega)\},$$

then there is a set Ω_0 *of full measure such that for all* $\omega \in \Omega_0$:

1. $M_t(\omega)$ *is open in* M *for each* $t > 0$, *i.e.* $\xi(\cdot,\omega)$ *is lower semicontinuous.*
2. $F_t(\cdot,\omega) : M_t(\omega) \to M$ *is in* C^{r-1} *and is a diffeomorphism onto an open subset of* M. *Moreover the map* : $t \mapsto F_t(\cdot,\omega)$ *is continuous into* $C^{r-1}(M_t(\omega))$, *with the topology of uniform convergence on compacta of the first r-1 derivatives.*
3. *Let* K *be a compact set and* $\xi^K = \inf_{x\in K} \xi(x)$. *Then*

$$\lim_{t \nearrow \xi^K(\omega)} \sup_{x\in K} d(x_0, F_t(x)) = \infty \tag{5.1}$$

almost surely on the set $\{\xi^K < \infty\}$. (Here x_0 is a fixed point of M and d is any complete metric on M.)

Remark: (1). For each compact set K, $\xi^K > 0$ almost surely. This is easily seen from $\xi(x) > 0$ a.s. for each x and the fact that a lower semicontinuous function on a compact set is bounded from below and assumes its minimum.

(2). As pointed out in [32], if there are two partial flows (F_t^1, ξ_1), and (F_t^2, ξ_2) which satisfy conditions 1-3 of theorem 5.1.1 for the C^0 topology, then we have uniqueness: for all x, $\xi_1(x) = \xi_2(x)$ almost surely. Consequently $\inf_{x\in M} \xi_1 = \inf_{x\in M} \xi_2$ almostly surely, and for each compact set K, $\xi_1^K = \xi_2^K$ almost surely. In particular if we have a version of the solution which is jointly continuous, then we actually have a version smooth in the C^{r-1} topology by part 3 of the theorem above.

From the theorem we see that starting from a compact set, the solution can be chosen to be continuous until part of it explodes. This and the following example suggests that strong completeness is a very demanding property, and there is a rich layer between being complete and being strongly complete.

Example: [30], [31] Let $X(x)(e) = e$, and $A = 0$. Consider the following stochastic differential equation $dx_t = dB_t$ on R^n-$\{0\}$ for $n > 1$. The solution is: $F_t(x) = x + B_t$, which is complete since for a fixed starting point x, $F_t(x)$ almost surely never hits 0 . But it is not strongly complete. However for any n-2 dimensional hyperplane (or a submanifold) H in the manifold, $\inf_{x\in H} \xi(x,\omega) = \infty$ a.s., since a Brownian motion does not charge a set of codimension 2. To get an example on a complete metric space, apply the inversion map $z \mapsto \frac{1}{z}$ in complex form as in [10]. The resulting system on R^2 is $(\tilde{X}, B)$ where

$$\tilde{X}(x,y) = \begin{bmatrix} y^2 - x^2 & 2xy \\ -2xy & y^2 - x^2 \end{bmatrix}.$$

The corresponding solution is in fact (in complex notation): $\frac{z}{1+zB_t}$. We'll continue this example on page 75.

This leads to the following definition suggested to me by D. Elworthy:

Definition 5.1.1 *A stochastic dynamical system on a manifold is called strongly p-complete if $\xi^K = \infty$ a.s. for every $K \in S_p$.*

Here S_p is the space of images of all smooth (smooth in the sense of extending over an open neighborhood) singular p-simplices. Recall a singular p-simplex in M is a map from the standard p-simplex to M. For convenience we also use the term singular p-simplex for the image of a singular p-simplex map.

The example above on R^n-$\{0\}$ (for $n > 2$) gives us a s.d.s. which is strongly n-2 complete, but not strongly (n-1)-complete. It is strongly (n-2)-complete since a singular (n-2)-simplex has finite Hausdorff (n-2) measure and is thus not charged by Brownian motion. It is not strongly (n-1) complete from proposition 5.2.3 on page 71.

Of all these "completeness" notions, we are particularly interested in strong 1-completeness, which helps us to get a result on $d(P_t f) = (\delta P_t) f$ (see page 92) and is used to get a homotopy vanishing result in theorem 7.3.2 replacing the obvious requirement of strong completeness. It turns out on most occasions, we only need strong 1-completeness and this follows from natural assumptions.

Once we get strong completeness, naturally we would like to know when does the flow *consist of diffeomorphisms*, i.e. there is a version of the flow such that except for a set of probability zero, $F_t(\cdot, \omega)$ is a diffeomorphism from M to M for each t and ω. This is basically the "onto" property of the flow, since the flow is always injective as showed in [52] and by part 2 of theorem 5.1.1. We will discuss this at the end of the next section.

Note: Results in this chapter remain true when (1.1) is changed to a time dependent equations.

5.2 Main Results

If not specified, by (F_t, ξ) we mean the partial flow defined in theorem 5.1.1.

Proposition 5.2.1 *If the stochastic differential equation considered is strongly p-complete, then $\xi^N = \infty$ a.s. for any p dimensional submanifold N of M.*

Proof: Let N be a p dimensional submanifold. Since all smooth differential manifolds have a smooth triangulation [55], we can write: $N = \cup V_i$. Here V_i are smooth singular p-simplexes. But $\xi^{V_i} = \infty$ a.s. for each i from the assumption. Thus $F.(\cdot)|_{V_i}$ is continuous a.s. and thus so is $F|_N$ itself. Thus $\xi^N = \infty$ a.s. from remark 2 on page 68. ■

Note if p equals the dimension of M, p-completeness gives back the usual definition of strong completeness, i.e. the partial flow defined in theorem 5.1.1 satisfies $\inf_{x \in M} \xi(x) = \infty$ almost surely as showed above. See also remark 2 after theorem 5.1.1. In this case we will continue to use strong completeness for strong n-completeness.

We need the following cocycle property from [31] for the next proposition: For almost all ω, for all $s > 0$, $t > 0$,

$$F_{t+s}(x, \omega) = F_t(F_s(x, \omega), \theta_s(\omega)). \tag{5.2}$$

Note that the exceptional set for (5.2) can be taken independent of s and t, according to a recent survey by L. Arnold (manuscript). However we do not need this refinement here.

Proposition 5.2.2 *Let $\xi = \inf_{x\in M} \xi(x)$. If there is a number $\delta > 0$ such that: $P\{\xi \geq \delta\} = 1$, then $\xi = \infty$ almost surely.*

Proof: There is no explosion by corollary 4.2.5. Let Ω_0 be the set of ω such that if $\omega \notin \Omega_0$, then $(t,x) \mapsto F_t(x,\omega)$ is continuous on $[0,\delta] \times M$. Consider $F_\cdot(\cdot, \theta_\delta(\omega))$ which is continuous on $[0,\delta] \times M$ if $\theta_\delta(\omega) \notin \Omega_0$. Let $\Omega_1 = \Omega_0 \cup \{\omega : \theta_\delta(\omega) \in \Omega_0\}$. Thus $\xi \geq 2\delta$ for $\omega \notin \Omega_1$ and Ω_1 has measure zero. Inductively we get $\xi = \infty$ almostly surely. ∎

The following proof was suggested to me by D. Elworthy, improving an earlier result proved in more restrictive situation:

Proposition 5.2.3 *A stochastic dynamical system on a n-dimensional manifold is strongly complete if strongly (n-1)-complete.*

Proof: Since strong n-completeness holds for compact manifolds, we shall assume M is not compact. Let B be a geodesic ball centered at some point p in M with radius smaller than the injectivity radius at p. Since M can be covered by a countable number of such balls, we only need to prove $\xi^B = \infty$ almost surely.

Let B be such a ball. It clearly divides M into two parts, one bounded and the other unbounded. Write $M - \partial B = K_0 \cup N_0$. Here K_0 is the bounded piece. Fix $T > 0$. By the ambient isotopy theorem there is a diffeomorphism from $[0,T] \times M$ to $[0,T] \times M$ given by: $(t,x) \mapsto (t, h_t(x))$ for h_t some diffeomorphism from M to its image, and satisfying:

$$h_t|_{\partial B} = F_t|_{\partial B}.$$

Set $K_t = h_t(K_0)$, $N_t = h_t(N_0)$. Then

$$M = K_t \cup F_t(\partial B) \cup h_t(N_0),$$

and

$$F_t(\mathring{B}) \subset K_t \tag{5.3}$$

on $\{\omega : t < \xi^B(\omega)\}$.

Now

$$\cup_{0\leq t\leq T} K_t = Proj^1 \left[H(K_0 \times [0,T])\right],$$

here Proj^1 denotes the projection to M. Thus $\cup_{0\leq t\leq T} K_t$ is compact. By (5.3), $F_t(B) = F_t(K_0) \cup F_t(\partial B)$, for $0 \leq t \leq T \wedge \xi^B$, stays in a compact region. So $\xi^B \geq T$ almost surely from part 3 of theorem 5.1.1. ∎

Take a sequence of nested relatively compact open sets $\{U_i\}$ such that it is a cover for M and $U_i \subset U_{i+1}$. Let λ^i be a standard smooth cut off function such that:

$$\lambda^i = \begin{cases} 1 & x \in U_{i+1} \\ 0, & x \notin U_{i+2}. \end{cases}$$

Let $X^i = \lambda^i X$, $A^i = \lambda^i A$, and F^i the solution flow to the s.d.s. (X^i, A^i). Then F^i can be taken smooth since both X^i and A^i have compact support. Let $S_i(x)$ be the first exit time of $F_t^i(x)$ from U_i and $S_i^K = \inf_{x\in K} S_i(x)$ for a compact set K. Thus S_i^K is a stopping time. Furthermore $F_t^i(x) = F_t(x)$ before S_i^K.

Clearly $S_i^K \leq \xi^K$, and in fact $\lim_{i\to\infty} S_i^K = \xi^K$ as proved in [10].

Let

$$K_1^1 = \{\text{Image}(\sigma) | \sigma : [0, \ell] \to M \text{ is } C^1,\ \ell < \infty\}.$$

Theorem 5.2.4 *Let M be a complete connected Riemannian manifold. Suppose all the coefficients of the stochastic differential equation are C^2, and assume there is a point $x \in M$ with $\xi(x) = \infty$ almost surely. Then we have $\xi^H = \infty$ for all $H \in K_1^1$, if*

$$\lim_{j\to\infty} \sup_{x\in K} E\left(|T_x F_{S_j^K}|\chi_{S_j^K<t}\right) < \infty \tag{5.4}$$

for every compact set $K \in K_1^1$ *and each* $t > 0$. *In particular when (5.4) holds we have strong 1-completeness, and strong completeness if dimension of* M *is less or equal to* 2.

proof: Let $y_0 \in M$. We shall show $\xi(y_0) = \infty$. Take a piecewise C^1 curve σ_0 connecting the two points x and y_0. Suppose it is parametrized by arc length σ_0: $[0, \ell_0] \to M$ with $\sigma_0(0) = x$.

Denote by K_0 the image set of the curve. Let $K_t = \{F_t(x) : x \in K_0\}$, and $\sigma_t = F_t \circ \sigma_0$. Then on $\{\omega : t < \xi^{K_0}(\omega)\}$, $\sigma_t(\omega)$ is a piecewise C^1 curve. Denote by $\ell(\sigma_t)$ the length of σ_t.

Let T be a stopping time such that $T < \xi^{K_0}$, then:

$$\begin{aligned}
\ell(\sigma_{T(\omega)}(\omega)) &\leq \int_0^{\ell_0} |\frac{d}{ds}\left(F_{T(\omega)}(\sigma(s),\omega)\right)| \, ds \\
&\leq \int_0^{\ell_0} |T_{\sigma(s)} F_T(\omega)| \, ds.
\end{aligned}$$

Thus for each $t > 0$:

$$E\ell(\sigma_T)\chi_{T<t} \leq \int_0^{\ell_0} E(\chi_{T<t}|T_{\sigma(s)}F_T|) \, ds \tag{5.5}$$

$$\leq \ell_0 \sup_{x\in K_0} E\left(|T_x F_T|\chi_{T<t}\right). \tag{5.6}$$

Assume $\xi^{K_0} < \infty$. Take T_0 with $P\{\xi^{K_0} < T_0\} > 0$. Now $\cup_{0\leq t\leq T_0} F_t(x,\omega)$ is a bounded set a.s. since $F_t(x)$ is sample continuous in t and $\xi(x) = \infty$. Thus there is $R(\omega) < \infty$ a.s. such that:

$$\sup_{0\leq t\leq T_0} d(F_t(x,\omega), x) \leq R(\omega) < \infty. \tag{5.7}$$

But by theorem 5.1.1, almostly surely on $\{\xi^{K_0} < \infty\}$

$$\lim_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(x, F_t(x,\omega)) = \infty. \tag{5.8}$$

So by the triangle inequality we get:

$$\sup_{x \in K_0} d(F_t(x,\omega), F_t(\bar{x},\omega)) \geq \sup_{x \in K_0} d(F_t(x,\omega), \bar{x}) - d(\bar{x}, F_t(\bar{x},\omega)).$$

Combining (5.7) with (5.8), we get, almostly surely on $\{\omega : \xi^{K_0} < T_0\}$:

$$\begin{aligned}
& \underline{\lim}_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(F_t(x,\omega), F_t(\bar{x},\omega)) \\
\geq & \underline{\lim}_{t \nearrow \xi^{K_0}} \sup_{x \in K_0} d(F_t(x,\omega), \bar{x}) - \sup_{0 \leq t \leq T_0} d(\bar{x}, F_t(\bar{x},\omega)) \\
= & \infty.
\end{aligned}$$

Therefore $\underline{\lim}_{t \nearrow \xi^{K_0}} \ell(\sigma_t(\omega)) = \infty$ almostly surely on $\{\xi^{K_0} < T_0\}$.

Let $T_j =: S_j^{K_0}$ be the stopping times defined immediately before this theorem, which converge to ξ^{K_0}. Then there is a subsequence of $\{T_j\}$, still denote by $\{T_j\}$, such that:

$$\underline{\lim}_{j \to \infty} \ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} = \infty,$$

Thus

$$E \underline{\lim}_{j \to \infty} \ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} = \infty.$$

However from equation (5.6) and our hypothesis (5.4) , we have:

$$\underline{\lim}_{j \to \infty} E\ell(\sigma_{T_j(\omega)}(\omega)) \chi_{\xi^{K_0} < T_0} \leq \ell_0 \underline{\lim}_{j \to \infty} \sup_{x \in K_0} E|T_x F_{T_j}| \chi_{T_j < T_0} < \infty$$

since $T_j < \xi^{K_0}$ almost surely. Applying Fatou's lemma, we have:

$$E \varliminf_{j \to \infty} \ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} \le \varliminf_{j \nearrow \infty} E\ell(\sigma_{T_j}) \chi_{\xi^{K_0} < T_0} < \infty.$$

This gives a contradiction. Thus $\xi^{K_0} = \infty$. In particular $\xi(y) = \infty$ for all $y \in M$.

Next take $K \in K_1^1$. Replacing K_0 by K in the proof above we get $\xi^K = \infty$. This is because in the proof above we only used the fact that there is a point x in K_0 with $\xi(x) = \infty$ and $|TF_t|$ satisfies (5.4).

To see strong 1-completeness, just notice the set of smooth singular 1-simplexes S_1 is contained in K_1^1. The proof is finished. ■

Remark:

(1). We only need inequality 5.4 to hold for one sequence of exhausting open sets $\{U_j\}$. In particular they may be different for different compact sets K.

(2). The second inequality holds if we assume:

$$\sup_{x \in K} E\left(\sup_{s \le t} |T_x F_s| \chi_{s < \xi(x)}\right) < \infty$$

for each number t and compact set K. However we keep the inequality with stopping times since it is sometimes easier to calculate from the original stochastic differential equation, and gives sharper result as will be shown later in the examples. See lemma 5.3.4 on page 84.

(3). The requirement on the connectness can be removed by assuming that the s.d.e. is complete at one point of each component of the manifold.

Example: (1). The requirement for the manifold to be complete is necessary. e.g. the example on $R^2 - \{0\}$ on page 68 satisfies equation (5.4) but is not strongly complete (see also the example on page 120). In fact the transformed flow $F_t(z) = \frac{z}{1+zB_t}$ on R^2 by inverting does not satisfy the condition of the theorem on its derivative and it is not strongly 1-complete.

(2). Theorem 5.2.4 does not work with equation (5.4) replaced by

$\sup_x E|T_xF_t| < \infty$. This can be seen by using the above example on $M = R^2 - \{0\}$ but with the following Riemannian metric:

$$|v|^{\sim} = \frac{|v|}{|x|}, \quad v \in T_xM,$$

since the metric is complete and for each compact set K,

$$\sup_{x\in K} E|T_xF_t|^{\sim} = \sup_{x\in K} E\frac{1}{|x+B_t|} < \infty.$$

We say a s.d.e. is *complete at one point* if there is a point x_0 in M with $\xi(x_0) = \infty$. From the theorem we have the following corollary, which is known for elliptic diffusions.

Corollary 5.2.5 *A stochastic differential equation with C^2 coefficients and satisfying hypothesis* (5.4) *in theorem* 5.2.4 *is complete if it is complete at one point.*

With strong 1-completeness we may apply the following Kolmogorov's criterion on regularity [31] to get strong p-completeness:

Kolmogorov's criterion: Let M be a complete Riemannian manifold with $d(,)$ denoting the distance between two points. Let F be a set of M-valued random variables indexed by $[0,1]^p$ for which there exist positive numbers γ, c, and ϵ such that:

$$Ed(F(\underline{s_1}), F(\underline{s_2}))^{\gamma} \leq c|\underline{s_1} - \underline{s_2}|^{p+\epsilon}$$

holds for all $\underline{s_1}, \underline{s_2}$ in $[0,1]^p$. Then there is a modification $\tilde{F}$ of F such that the paths of $\tilde{F}$ are continuous. Here $|\underline{s_1} - \underline{s_2}|$ is the distance between these two points (induced from R^p).

There is a corresponding result for pathwise continuous stochastic processes $\{F_t(\cdot), t \geq 0\}$ parametrized by $[0,1]^p$. There is a version which is jointly continuous in t and x on $[0,T] \times [0,1]^p$ for each T, if the following is satisfied:

$$E \sup_{s \leq t} d(F_s(\underline{s_1}), F_s(\underline{s_2}))^\gamma \leq c|\underline{s_1} - \underline{s_2}|^{p+\epsilon}$$

for all $\underline{s_1}, \underline{s_2}$ in $[0,1]^p$. This comes naturally by letting $N = C([0,T], M)$ with the following metric: $d(f,g) = \sup_{0 \leq s \leq T} d(f(s), g(s))$.

Recall a map $\alpha : [0,1]^p \to M$ is called Lipschitz continuous if there is a constant c such that:

$$d(\alpha(\underline{s}), \alpha(\underline{t})) \leq c|\underline{s} - \underline{t}| \tag{5.9}$$

for all $\underline{s}$, $\underline{t}$ in $[0,1]^p$.

Denote by L_p the space of all the image sets of such a Lipschitz map. This space contains K_p.

Theorem 5.2.6 *Let M be a complete connected Riemannian manifold. Consider a s.d.e. which is complete at one point and with C^2 coefficients. Let $1 \leq d \leq n$. Then we have $\xi^K = \infty$ for each $K \in L_d$ if for each positive number t and compact set K there is a number $\delta > 0$ such that:*

$$\sup_{x \in K} E\left(\sup_{s \leq t} |T_x F_s|^{d+\delta} \chi_{s<\xi}\right) < \infty.$$

In particular this implies strong d-completeness.

Proof: Let σ be a Lipschitz map from $[0,1]^d$ to M with image set K. Take a compact set $\tilde{K}$ with the following property: for any two points of K, there is a piecewise C^1 curve lying in $\tilde{K}$ connecting them.

For example the set $\tilde{K}$ can be taken in the following way: Let σ be a minimum length curve in M connecting two points of K; its length will be smaller or equal to $\text{dia}(K)$. Let $\tilde{K}$ be the closure of the union of the image sets of such curves.

Let $x = \sigma(\underline{s})$ and $y = \sigma(\underline{t})$ be two points from K. Let α be a piecewise C^1 curve in $\tilde{K}$ connecting them. Denote by H_α the image set of α and ℓ its length. By proposition 5.2.4, $\xi^{H_\alpha} = \infty$. Thus for any $T_0 > 0$ we have:

$$\begin{aligned}
E \sup_{t \leq T_0} [d(F_t(x), F_t(y))]^{d+\delta} &\leq E\left(\int_0^\ell \sup_{t\leq T_0} |T_{\alpha(s)}F_t|\, ds\right)^{d+\delta} \\
&\leq \ell^{d+\delta-1} E \int_0^\ell \left(\sup_{t\leq T_0} |T_{\alpha(s)}F_t|^{d+\delta}\right) ds \\
&\leq \ell^{d+\delta} \sup_{x\in\tilde{K}} \left(E \sup_{t\leq T_0} |T_x F_t|^{d+\delta}\right).
\end{aligned}$$

Taking infimum over a sequence of such curves which minimizing the distance between x and y we get:

$$E\left(\sup_{t\leq T_0} d(F_t(x), F_t(y))^{d+\delta}\right) \leq d(x,y)^{d+\delta} \sup_{x\in\tilde{K}} E\left(\sup_{t\leq T_0} |T_x F_t|^{d+\delta}\right).$$

The Lipschitz property of the map σ gives

$$E\left(\sup_{t\leq T_0} d(F_t(\sigma(\underline{s}), F_t(\sigma(\underline{t})))^{d+\delta}\right) \leq c|\underline{s} - \underline{t}|^{d+\delta} \sup_{x\in\tilde{K}} E\left(\sup_{t\leq T_0} |T_x F_t|^{d+\delta}\right).$$

Thus we have a modification of $\tilde{F}_\cdot(\sigma(-))$ of $F_\cdot(\sigma(-))$ which is jointly continuous from $[0, T_0] \times [0,1]^d \to M$, according to the Kolmogorov's criterion. So for a fixed point x_0 in M:

$$\sup_{t\in[0,T_0]} \sup_{\underline{s}\in[0,1]^d} d(F_t(\sigma(\underline{s}),\omega),x_0) < \infty.$$

On the other hand on $\{\xi^K < \infty\}$,

$$\lim_{t\nearrow\xi^K} \sup_{x\in K} d(F_t(x,\omega),x_0) = \infty$$

almost surely. This give a contradiction. So $\xi^K = \infty$ for all $K \in L_d$.

Notice every singular d-simplex has a representation of a Lipschitz map from the cube $[0,1]^d$ to M (by squashing one half of the cube to the diagonal). This gives the required strong p-completeness. ■

This theorem is used in section 7.3 to get a cohomology vanishing result.

Flows of diffeomorphisms

To look at the diffeomorphism property, we first quote the following theorem (first proved by Kunita) from [10]:

Let $M = R^n$. Assume both X and A are C^2. If the s.d.s. is strongly complete, then it has a flow which is surjective for each $T > 0$ with probability one if and only if the adjoint system:

$$dy_t = X(y_t) \circ dB_t - A(y_t)dt \tag{5.10}$$

is strongly complete.

For a manifold this works equally well since equation (5.10) does give the inverse map up to distribution. Thus if both equations (1.1) and (5.10) satisfy the conditions of the theorem for strong completeness, the solution flow F_t consists of diffeomorphisms. When there is a uniform cover for (X, A), there is no explosion for both (1.1) and (5.10) . In this case, the solution consists of diffeomorphisms if for K compact:

$$\sup_{x\in K} E \sup_{s\le t}\left(|T_xF_s|^{n-1+\delta} + \left(|T_{F_s^{-1}(x)}F_s|^{-1}\right)^{n-1+\delta}\right) < \infty,$$

since both (1.1) and (5.10) are strongly complete by theorem 5.2.6.

5.3 Applications

In this section we look at the stochastic differential equation to see when the conditions for the theorems above are fulfilled.

As before let M be a Riemannian manifold. Let $x_0 \in M$, $v_0 \in T_{x_0}M$. Noticing $|v_t|$ is almost surely nonzero for all t, [52], we have the following formula for the p^{th} power of the norm of v_t from [26] for all p:

$$\begin{aligned} |v_t|^p = \ & |v_0|^p + p\sum_{i=1}^m \int_0^t |v_s|^{p-2} < \nabla X^i(v_s), v_s > dB_s^i \\ & + p\int_0^t |v_s|^{p-2} < \nabla A(v_s), v_s > ds \\ & + \frac{p}{2}\sum_1^m \int_0^t |v_s|^{p-2} < \nabla^2 X^i(X^i, v_s), v_s > ds \\ & + \frac{p}{2}\sum_1^m \int_0^t |v_s|^{p-2} < \nabla X^i(\nabla X^i(v_s)), v_s > ds \\ & + \frac{p}{2}\sum_1^m \int_0^t |v_s|^{p-2} < \nabla X^i(v_s), \nabla X^i(v_s) > ds \\ & + \frac{1}{2}p(p-2)\sum_1^m \int_0^t |v_s|^{p-4} < \nabla X^i(v_s), v_s >^2 ds. \end{aligned} \tag{5.11}$$

on $\{t < \xi\}$. Let $v \in T_xM$. Define $H_p(v,v)$ as follows:

$$\begin{aligned} H_p(v,v) = \ & 2 < \nabla A(x)(v), v > + \sum_{i=1}^m < \nabla^2 X^i(X^i, v), v > \\ & + \sum_1^m < \nabla X^i(\nabla X^i(v)), v > + \sum_1^m < \nabla X^i(v), \nabla X^i(v) > \\ & + (p-2)\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2 . \end{aligned}$$

Let τ be a stopping time, then

$$\begin{aligned} |v_{t\wedge\tau}|^p = \ & |v_0|^p + p\sum_{i=1}^m \int_0^{t\wedge\tau} |v_s|^{p-2} < \nabla X^i(v_s), v_s > dB_s^i \\ & + \frac{p}{2}\int_0^{t\wedge\tau} |v_s|^{p-2} H_p(v_s, v_s) ds. \end{aligned} \tag{5.12}$$

This gives:

$$
\begin{aligned}
|v_{t\wedge\tau}|^{2p} &\leq 2|v_0|^{2p} + 4p^2\left[\sum_1^m \int_0^{t\wedge\tau} |v_s|^{p-2} < \nabla X^i(v_s), v_s > dB_s^i\right]^2 \\
&\quad +4p^2\left[\int_0^{t\wedge\tau} |v_s|^{p-2} H_p(v_s, v_s) ds\right]^2 \\
&\leq 2|v_0|^{2p} + 4p^2 2^{m-1}\sum_1^m \left[\int_0^{t\wedge\tau} |v_s|^{p-2} < \nabla X^i(v_s), v_s > dB_s^i\right]^2 \\
&\quad +4p^2\left[\int_0^{t\wedge\tau} |v_s|^{p-2} H_p(v_s, v_s) ds\right]^2.
\end{aligned}
$$

Let T be a positive number, then:

$$
\begin{aligned}
E\sup_{t\leq T} |v_{t\wedge\tau}|^{2p} &\leq 2|v_0|^{2p} \\
&\quad +2^{m+1}p^2 \textstyle\sum_1^m E\sup_{t\leq T}\left[\int_0^{t\wedge\tau} |v_s|^{p-2} < \nabla X^i(v_s), v_s > dB_s^i\right]^2 \\
&\quad +4p^2 E\sup_{t\leq T}\left[\int_0^{t\wedge\tau} |v_s|^{p-2} H_p(v_s, v_s) ds\right]^2.
\end{aligned}
$$

Applying Burkholder-Davies-Gundy inequality and Hölder's inequality we get:

$$
\begin{aligned}
E\sup_{t\leq T} |v_{t\wedge\tau}|^{2p} &\leq 2|v_0|^{2p} + 4Tp^2 E\int_0^T \chi_{s\leq\tau}|v_s|^{2p-4} H_p^2(v_s, v_s) ds \\
&\quad +2^{m+1} m p^2 c_0 E\int_0^T \chi_{s\leq\tau}|v_s|^{2p-4} < \nabla X^i(v_s), v_s >^2 ds.
\end{aligned}
$$

Here c_0 is the constant in Burkholder's inequality.

Let U be a relatively compact open set. Denote by $\tau(x)$ the first exit time of $F_t(x)$ from U. For simplicity we write τ instead of $\tau(x_0)$.

Since X and A are C^2, there is a constant c such that: $|\nabla X^i(x_s)|^2 < c$ and $|H_p(v, v)| < c|v|^2$ on the set $\{s \leq \tau(x_0)\}$. Let $k_p = 4(cp)^2(T + 2^{m-1}mc_0)$, we have:

$$
\begin{aligned}
E \sup_{t \leq T} |v_{t\wedge\tau}|^{2p} &\leq 2|v_0|^{2p} + 2^{m+1} m p^2 c_0 E \int_0^T c^2 |v_s|^{2p} \chi_{s\leq\tau} ds \\
&\quad + 4Tp^2 E \int_0^T c^2 |v_s|^{2p} \chi_{s\leq\tau} ds \\
&= 2|v_0|^{2p} + k_p E \int_0^T |v_s|^{2p} \chi_{s\leq\tau} ds \\
&\leq 2|v_0|^{2p} + k_p E \int_0^T E\left(\sup_{u\leq s} |v_u|^{2p} \chi_{u\leq\tau} \right) ds \\
&\leq 2|v_0|^{2p} + k_p E \int_0^T E\left(\sup_{u\leq s} |v_{u\wedge\tau}|^{2p} \chi_{u\leq\tau} \right) ds \\
&\leq 2|v_0|^{2p} + k_p E \int_0^T E\left(\sup_{u\leq s} |v_{u\wedge\tau}|^{2p} \right) ds.
\end{aligned}
$$

By Gronwall's lemma:

$$
E \sup_{t\leq T} |v_{t\wedge\tau}|^{2p} \leq 2|v_0|^{2p} e^{k_p T}.
$$

On the other hand, taking an orthonormal basis $\{e_i\}_1^n$ of $T_{x_0}M$, we have:

$$
E\left(\sup_{t\leq T} |T_{x_0} F_{t\wedge\tau}|^{2p} \right) \leq c \sum_1^n E\left(\sup_{t\leq T} |T_{x_0} F_{t\wedge\tau}(e_i)|^{2p} \right) \leq c e^{k_p T}.
$$

Here c denotes some constant depending only on p and n. Thus we arrived at the following useful lemma, which is fairly well known.

Lemma 5.3.1 *(1). For each relatively compact open set U, there is a constant c depending only on the bounds of the coefficients of the stochastic differential equation on U such that for all p:*

$$
E \sup_{s\leq t} |T_x F_{s\wedge\tau(x)}|^{2p} \leq 2c_1 e^{cp^2 t}. \tag{5.13}
$$

Here c_1 is a constant depends on $n = dim(M)$.

(2). Assume both $|\nabla X|$ and $|H_p|$ are bounded. By the latter we mean $|H_p(v,v)| \le k|v|^2$ for some constant k and all $v \in T_xM$. Then

$$\sup_{x\in M} E\left(\sup_{s\le t} |T_xF_{s\wedge\xi}|\right)^{2p} \le 2c_1 e^{cp^2t} \tag{5.14}$$

for all p. Here c is a constant and c_1 depends only on $n = dim(M)$. In particular this is the case if X, ∇X, $\nabla^2 X$, and ∇A are all bounded.

For the proof of (5.14), let $\tau = \tau^{U_n}$ in the above calculation, for $\{U_n\}$ a sequence of nested relatively compact open set exhausting M. Then take the limit. ∎

As a corollary, we have the following result on strong completeness:

Corollary 5.3.2 *Let M be a complete connected Riemannian manifold. Assume $|\nabla X|$ is bounded and there is a constant k such that $|H_1(v,v)| < k|v|^2$. Then the s.d.s. is strongly complete if complete for one point. Note the last condition is satisfied if $|X| + |\nabla X| + |\nabla^2 X| + |\nabla A|$ is bounded.*

Note we do not use any sort of nondegeneracy in the above corollary.

Let $M = R^n$. Assume the s.d.e. is given in Itô form. We have as a corollary the following known result:

Corollary 5.3.3 *A stochastic differential equation on R^n (in Itô form) is strongly complete if all coefficients are C^2 and globally Lipschitz continuous.*

Proof: First we have completeness as is well known. Write:

$$\begin{aligned} dx_t &= X(x_t)dB_t + A(x_t)dt \\ &= X(x_t) \circ dB_t + A(x_t)dt. \end{aligned}$$

Here $\tilde{A} = A - \frac{1}{2}\sum_1^m \nabla X^i(X^i)$. So

$$\nabla \tilde{A} = \nabla A - \frac{1}{2}\sum_1^m \nabla^2 X^i(v, X^i) - \frac{1}{2}\sum_1^m \nabla X^i(\nabla X^i(v)).$$

Note also on R^n: $\nabla^2 X^i(X^i, v) = \nabla^2 X^i(v, X^i)$ on R^n. Substituting these in equation 5.12 the second derivatives of X^i disappear. Thus

$$\begin{aligned} H_p(v,v) = \; & 2 < \nabla \tilde{A}(v), v > + \textstyle\sum_1^m < \nabla X^i(v), \nabla X^i(v) > \\ & +(p-2)\textstyle\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2 . \end{aligned}$$

So the boundedness of ∇X and ∇A give us strong completeness. ■

Lemma 5.3.4 *Let $\{U_j\}$ be a sequence of relatively compact open sets exhausting M, and S_j^k the stopping times defined before theorem* 5.2.4 *on page* 72. *Here K is a compact set.*

Assume $H_p(v,v) \le k|v|^2$ for some constant k. Then for all j

$$\sup_M E(T_x F_{t\wedge S_j^K}|^p) \le e^{\frac{p}{2}kt}. \tag{5.15}$$

Proof: First we have: $E\sup_{s<t}|TF_{s\wedge S_j^K}|^{2p} < \infty$ for $x \in K$ by a similar proof as for (5.13). So from (5.12) on page 80, we obtain:

$$E|v_{t\wedge S_j^K}|^p = |v_0|^p + \frac{p}{2} E\int_0^{t\wedge S_j^K} |v_s|^{p-2} H_p(v_s, v_s)\, ds,$$

since the martingale part disappears. Thus, just as before, there is the following estimate:

$$E|T_x F_{t\wedge S_j^K}|^p \le e^{\frac{p}{2}kt}$$

from $H_p(v,v) \le k|v|^2$ and Gronwall's lemma. ■

Remark: In fact (5.15) holds if S_j^K is replaced by $S_j(x)$. Here $S_j(x)$ is the first exit time of $F_t(x)$ from U_j.

From the proof, we have the following corollary of theorem 5.2.4:

Corollary 5.3.5 *A s.d.e. is strongly 1-complete if it is complete at one point and satisfies:*

$$H_1(v,v) \le k|v|^2.$$

Here k is a constant.

The case of Brownian system

Next we consider special cases. First we assume the s.d.s. considered is a Brownian motion with drift Z. Then $Z = \frac{1}{2}\sum_1^m \nabla X^i(X^i) + A$, and

$$\nabla Z(v) = \frac{1}{2}\sum_1^m \nabla^2 X^i(v, X^i) + \frac{1}{2}\sum_1^m \nabla X^i\left(\nabla X^i(v)\right) + \nabla A(v).$$

On the other hand,

$$\begin{aligned} \left\langle \nabla^2 X^i(X^i, v), v\right\rangle - \left\langle \nabla^2 X^i(v, X^i), v\right\rangle &= \left\langle R(X^i, v)(X^i), v\right\rangle \\ &= -\mathrm{Ric}(v,v). \end{aligned}$$

Here R is the curvature tensor and Ric is the Ricci curvature. Thus:

$$\begin{aligned} H_p(v,v) = \quad & 2 < \nabla Z(x)(v), v > -Ric_x(v,v) + \textstyle\sum_1^m |\nabla X^i(v)|^2 \\ & +(p-2)\textstyle\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2 . \end{aligned} \tag{5.16}$$

And therefore we have the following theorem:

Theorem 5.3.6 *Assume the s.d.s. is a Brownian motion with gradient drift ∇h. Then if $\frac{1}{2}\mathrm{Ric} - Hess(h)$ is bounded from below with $|\nabla X|$ bounded the Brownian motion is strongly complete. Here $Hess(h) = \nabla^2 h$.*

proof: By a result in [5], we have completeness if $\frac{1}{2}$ Ricc-Hess(h) is bounded from below. The strong completeness follows from theorem 5.2.6 and lemma 5.3.1. ■

In the above theorem, the s.d.s. may be a Brownian motion with a general drift if we know the system does not explode a priori. The nonexplosion problem is discussed in chapter 4 and chapter 7.

The case of gradient Brownian system

Next we consider gradient Brownian motion as in the introduction and follow the ideas of [28]. Let ν_x be the space of normal vectors to M at x. There is the second fundamental form:

$$\alpha_x : T_xM \times T_xM \to \nu_x$$

and the shape operator:

$$A_x : T_xM \times \nu_x \to T_xM$$

related by $\langle \alpha_x(v_1, v_2), w\rangle = \langle A_x(v_1, w), v_2\rangle$. If $Z(x) : R^m \to \nu_x$ is the orthogonal projection, then

$$\nabla X^i(v) = A_x(v, Z(x)e_i)$$

as showed in [31] and [25].

Let $f_1, \dots f_n$ be an o.n.b. for T_xM. Consider $\alpha_x(v, \cdot)$ as a linear map from T_xM to ν_x. Denote by $|\alpha_x(v, \cdot)|_{H,S}$ the corresponding Hilbert Schmidt norm, and $|\cdot|_{\nu_x}$ the norm of a vector in ν_x. Accordingly we have:

$$\sum_1^m \Big\langle \nabla X^i(v), \nabla X^i(v)\Big\rangle = \sum_{i=1}^m \sum_{j=1}^n \langle A_x(v, Z(x)e_i), f_j\rangle^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\langle \alpha_x(v,f_j), Z(x)e_i\rangle^2\\
&= \sum_{j=1}^{m}|\alpha_x(v,f_j)|^2_{\nu_x}\\
&= |\alpha_x(v,\cdot)|^2_{H.S}.
\end{aligned}
$$

There is also:

$$\sum_1^m \left\langle \nabla X^i(v), v\right\rangle^2 = |\alpha_x(v,v)|^2_{\nu_x},$$

giving

$$
\begin{aligned}
H_p(v,v) = \ & -\mathrm{Ric}(v,v) + 2 < \nabla Z(v), v > + |\alpha_x(v,\cdot)|^2_{H.S}\\
& + \tfrac{(p-2)}{|v|^2}|\alpha_x(v,v)|^2_{\nu_x}.
\end{aligned}
\tag{5.17}
$$

Thus the corollary:

Corollary 5.3.7 *Assume the second fundamental form is bounded. Then the gradient Brownian motion with drift ∇h is strongly complete if $\frac{1}{2}Ric - Hess(h)$ is bounded from below. It consists of diffeomorphisms if both $\frac{1}{2}Ric - Hess(h)$ and $\frac{1}{2}Ricci + Hess(h)$ are bounded from below.*

Proof: The strong completeness is clear from the previous theorem. The diffeomorphism property comes from the fact that its adjoint equation is also a gradient Brownian system (with drift -∇h).

Further, there is the following Gauss's theorem:

$$\mathrm{Ric}(v,v) = \langle \alpha(v,v), \mathrm{trace}\ \alpha\rangle - |\alpha(v,\cdot)|^2_{H.S}.$$

Giving:

$$
\begin{aligned}
H_p(v,v) = \ & - < \alpha(v,v), \mathrm{trace}\ \alpha > + 2|\alpha_x(v,\cdot)|^2_{H.S}\\
& + \tfrac{1}{|v|^2}(p-2)|\alpha_x(v,v)|^2_{\nu_x} + 2 < \mathrm{Hess}(h)(v), v > .
\end{aligned}
\tag{5.18}
$$

Thus the completeness and strongly completeness of a gradient Brownian motion rely only on the bound on the second fundamental form and the bound on the drift: there is no explosion if $h = 0$ and $|\alpha_x| \leq k(|x|)$ with k a function on R_+ satisfying:

$$\int \frac{1}{k(r)} dr = \infty$$

from Gauss theorem and the example on page 57 in chapter 4. Furthermore a gradient Brownian motion is strongly complete if the second fundamental form is bounded.

Chapter 6

Derivative semigroups

6.1 Introduction

Assume the derivative of the solution flow of equation (1.1) has first moment: $E|T_xF_s\chi_{s<\xi(x)}| < \infty$, we may define a semigroup (formally) of linear operators δP_t on 1-forms as follows: for $v \in T_xM$ and ϕ a 1-form

$$\delta P_t\phi(v) = E\phi\left(T_xF_t(v)\right)\chi_{t<\xi(x)} \tag{6.1}$$

It is in fact a semigroup on $L^\infty(\Omega)$, the space of bounded 1-forms, if

$$\sup_x E|T_xF_t\chi_{t<\xi(x)}| < \infty. \tag{6.2}$$

We are interested in three problems:

1. When is δP_t well defined, as a strongly continuous semigroup?

2. When is $dP_tf = \delta P_t(df)$?

3. When is $\delta P_t\phi = e^{\frac{1}{2}t\Delta^{h,1}}\phi$, if $\mathcal{A} = \frac{1}{2}\Delta^h$?

If all answers to the questions are yes, we can obtain informations of heat semigroups and answer the question wether δP_t sends closed forms to closed forms. These problems are also the basis for the next two chapters and will be discussed in detail and in great generality in this chapter. For related discussions, see Vauthier[66] and Elworthy[25].

However first let $\mathcal{A} = \frac{1}{2}\Delta^h$. We have nonexplosion if $dP_t f = \delta P_t(df)$ for $f \in C_K^\infty$ and if $E|T_x F_t| < \infty$ for all x, as will be shown in proposition 7.2.2 in chapter 7. So it is natural to assume completeness. For a complete stochastic dynamical system on a complete Riemannian manifold, two basic assumptions:

$$E \sup_{s \le t} |T_x F_s| < \infty \tag{6.3}$$

and

$$\sup_{x \in M} E|T_x F_t|^{1+\delta} < \infty \tag{6.4}$$

will give everything we need:

1. $\delta P_t \phi = e^{\frac{1}{2}t\Delta^{h,1}} \phi$, for $\phi \in L^\infty$,

2. $dP_t f = \delta P_t(df)$, for $f \in C_K^\infty$

3. and strong 1-completeness.

as shown later. c.f. theorem 5.2.4 on page 73, theorem 6.3.1 on page 92 and proposition 7.2.2 on page 108. This basic assumption is satisfied by solutions of a s.d.e. with all the coefficients and their first two derivatives bounded as shown in section 5.3.

But first we recall the properties of probability semigroups for functions.

6.2 Semigroups for functions

Let P_t be the probability semigroup on bounded measurable functions determined by our stochastic dynamical system. Let $\mathcal{A}$ be its generator. Then

$\mathcal{A} = \frac{1}{2}\sum X^iX^i + A$ on C_k^∞, the space of smooth compactly supported functions. If further we assume completeness, then it sends bounded continuous functions to bounded continuous functions as showed in [31].

Let $\mathcal{A} = \frac{1}{2}\Delta^h$, then P_t is a strongly continuous L^2 semigroup restricting to $L^2 \cap L^\infty$(see [33]). Associated with $\frac{1}{2}\Delta^h$, there is also the functional analytic semigroup $e^{\frac{1}{2}t\Delta^h}$. These two semigroups agree as in [33]. See also the proposition below. Thus from theorem 1.6.1 on page 23, P_tf is smooth on $L^2 \cap L^\infty$. Moreover P_t is L^p contractive on $L^2 \cap L^p \cap L^\infty$ for all t, $1 \leq p \leq \infty$. See [37] for more discussions on the L^p contractivity of probability semigroups.

Finally we have the following known result :

Proposition 6.2.1 *Let M be a complete Riemannian manifold, then*

$$P_t1(x) = e^{\frac{1}{2}t\Delta^h}1(x) = \int p_t^h(x,y)e^{h(y)}dy.$$

Proof: First we show $P_tf = e^{\frac{1}{2}t\Delta^h}f$ for $f \in L^2 \cap L^\infty$. Since P_t is a strongly continuous L^2 semigroup on $L^2 \cap L^\infty$, it extends to a strongly continuous L^2 semigroup $\bar{P}_t$ on the whole L^2 space. Let $\bar{\mathcal{A}}$ be the generator of $\bar{P}_t$. Then $\bar{\mathcal{A}}$ is a closed operator by theorem 1.4.1 and agrees with $\frac{1}{2}\Delta^h$ on C_K^∞. Thus $\bar{\mathcal{A}} = \frac{1}{2}\Delta^h$ since there is only one closed extension for $\frac{1}{2}\Delta^h$ from the essential self-adjointness of Δ^h obtained in chapter 2.

Applying the uniqueness theorem for the semigroup of class C_0(theorem 1.4.1), we get $\bar{P}_tf = e^{\frac{1}{2}t\Delta^h}f$. Thus $P_tf = e^{\frac{1}{2}t\Delta^{h,1}}$ on $L^2 \cap L^\infty$.

Next let $\{g_n\}$ be an increasing sequence of functions in C_K^∞ approaching 1 with $0 \leq g_n \leq 1$. Such a sequence exists as shown in the appendix. Then $e^{\frac{1}{2}t\Delta^h}g_n \to e^{\frac{1}{2}t\Delta^h}1$ since $p_t^h(x,-)$ is in L^1 (c.f. theorem 1.6.1). But $e^{\frac{1}{2}t\Delta^h}g_n(x) = P_tg_n(x) \to P_t1(x)$ for each x. So the limits must be equal: $P_t1 = e^{\frac{1}{2}t\Delta^h}1$. ■

6.3 δP_t and dP_t

With the help of results on strong 1-completeness, we have the following theorem which improves a theorem in [31], where strong completeness and bounds on curvatures are assumed.

Theorem 6.3.1 *Assume strong 1-completeness and suppose for each compact set K, there is a constant $\delta > 0$ such that:*

$$\sup_{x \in K} E|T_x F_t|^{1+\delta} < \infty.$$

Then $P_t f$ is C^1 and

$$d(P_t f) = (\delta P_t)(df)$$

for any C^1 function f with both f and df bounded.

Proof: Let $(x, v) \in TM$. Take a geodesic curve $\sigma: [0, \ell] \to M$ starting from x with velocity v. By the strong 1-completeness, $F_t(\sigma(s))$ is a.s. differentiable with respect to s. So for almostly all ω:

$$\frac{f\left(F_t\left(\sigma(s), \omega\right)\right) - f\left(F_t(x, \omega)\right)}{s} = \frac{1}{s} \int_0^s df\left(T_{\sigma(r)} F_t\left(\dot{\sigma}(r), \omega\right)\right) dr.$$

Let

$$I_s = \frac{1}{s} \int_0^s df\left(T_{\sigma(r)} F_s(\dot{\sigma}(r), \omega)\right) dr.$$

We want to show: $\lim_{s \to 0} EI_s = E \lim_{s \to 0} I_s$. By the strong 1-completeness, we know $T_{\sigma(r)} F_t(\dot{\sigma}(r))$ is continuous in r for almost all ω. Thus:

$$\begin{aligned} E \lim_{s \to 0} I_s &= E \lim_{s \to 0} \frac{1}{s} \int_0^s df(T_{\sigma(r)} F_t(\dot{\sigma}(r))(\omega) dr \\ &= E df(TF_t(v)). \end{aligned}$$

On the other hand, $Edf(T_{\sigma(r)}F_t(\dot\sigma(r)))$ is continuous in r if $|T_{\sigma(r)}F_t|$ is uniformly integrable in r with respect to the probability measure P. This is so if

$$\sup_r E|T_{\sigma(r)}F_t|^{1+\delta} < \infty.$$

Thus

$$\lim_{s\to 0}\frac{1}{s}\int_0^s Edf\left(T_{\sigma(r)}F_t(\dot\sigma(r),\omega)\right) dr = Edf(TF_t(v))$$

and the proof is finished. ■

Corollary 6.3.2 *Let M be a complete Riemannian manifold. Suppose our s.d.s. is complete. If*

$$E\sup_{s\le t}|T_xF_s| < \infty$$

and

$$\sup_{x\in K} E|T_xF_t|^{1+\delta} < \infty$$

for each $t > 0$, and K compact, then $dP_tf = (\delta P_t)(df)$ for all functions f with both f and df bounded.

In terms of the coefficients of s.d.e., we have:

Corollary 6.3.3 *Let M be a complete Riemannain manifold. Suppose our s.d.s. is complete and satisfies:*

$$H_{1+\delta}(v,v) \le k|v|^2.$$

Then $dP_tf = \delta P_t(df)$ if both f and df are bounded. Here

$$\begin{aligned} H_p(v,v) = \; & 2 < \nabla A(x)(v), v > + \textstyle\sum_{i=1}^m < \nabla^2 X^i(X^i, v), v > \\ & + \textstyle\sum_1^m < \nabla X^i(\nabla X^i(v)), v > + \sum_1^m < \nabla X^i(v), \nabla X^i(v) > \\ & + (p-2)\textstyle\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2 . \end{aligned}$$

Proof: From corollary 5.3.5 on page 85, we have strong 1-completeness. Furthermore

$$\sup_{x\in M} E|T_xF_t|^{\frac{1+\delta}{2}} < \infty$$

if $H_{1+\delta}(v,v) \le k|v|^2$. ■

Remark: For a stochastic differential equation on R^n (in Itô form),

$$\begin{aligned} H_p(v,v) = \; & 2 < \nabla A(x)(v), v > + \textstyle\sum_1^m < \nabla X^i(v), \nabla X^i(v) > \\ & +(p-2) \textstyle\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2, \end{aligned}$$

as on page 83. Also notice if

$$|X(x)| \le k(1+|x|)$$

and

$$< A(x), x > \le k(1+|x|)|x|,$$

then the system is complete. Thus our result on $dP_tf = \delta P_t(df)$ improves a theorem in [31], where global Lipschitz continuity is assumed of the coefficients. Here is a brief proof for the nonexplosion claim we made:

Let $x_0 \in M$, $x_s = F_s(x)$. On $\{\omega : t < \xi(x,\omega)\}$,

$$\begin{aligned} |x_t|^2 \;=\; & |x_0|^2 + 2\int_0^t < x_s, X(x_s)dB_s > +2\int_0^t < x_s, A(x_s) > ds \\ & + \int_0^t |X(x_s)|^2 ds. \end{aligned}$$

Let $T_n(x)$ be the first exit time of $F_t(x)$ from the ball $B_n(0)$. Write $T_n = T_n(x_0)$ for simplicity. Then

$$\begin{aligned} E|x_{t\wedge T_n}|^2 \;=\; & |x_0|^2 + 2E\int_0^{t\wedge T_n} < x_s, A(x_s) > ds + E\int_0^{t\wedge T_n} |X(x_s)|^2 ds \\ \le\; & |x_0|^2 + 6k\int_0^{t\wedge T_n} (1+|x_s|^2) ds. \end{aligned}$$

By Grownall's inequality:

$$E|x_{t\wedge T_n}|^2 \le (|x_0|^2 + 6kt)e^{6kt}.$$

Thus $P\{T_n < t\} = 0$, so there is no explosion.

Lemma 6.3.4 *Let M be a complete Riemannian manifold. Consider a s.d.s. with generator $\mathcal{A} = \frac{1}{2}\Delta^h$. Let $f \in C_K^\infty$. Then the following are equivalent:*

1. $e^{\frac{1}{2}t\Delta^{h,1}}(df) = \delta P_t(df)$.
2. $\delta P_t(df) = d(P_t f)$.

This is because $P_t f = e^{\frac{1}{2}t\Delta^h} f$ from section 6.2 and $e^{\frac{1}{2}t\Delta^{h,1}}(df) = d(e^{\frac{1}{2}t\Delta^h} f)$ from proposition 2.3.1.

Corollary 6.3.5 *Let M be a complete Riemannian manifold. Assume the conditions of the above theorem and $\mathcal{A} = \frac{1}{2}\Delta^h$. Then for $f \in C_K^\infty$,*

$$\delta P_t(df) = e^{\frac{1}{2}t\Delta^{h,1}}(df).$$

Remark: We will show later that if $d(P_t f) = \delta P_t(df)$ for f with f and df bounded, then there is no explosion. See proposition 7.2.2 and its corollary.

Finally as known in the compact case [25], we also have:

Proposition 6.3.6 *Assume $E\left(|T_x F_t|\chi_{t<\xi(x)}\right)$ is finite and continuous in t for $t \in [0, a]$. Here a is positive constant. Let ϕ be a 1-form in C_K^∞. Then*

$$\frac{\partial(\delta P_t \phi)}{\partial t}|_{t=0} = \mathcal{L}\phi$$

with $\lim_{t\to 0} \delta P_t \phi(v) = \phi(v)$ for each $v \in T_x M$. Here $\mathcal{L}$ is as defined on page 13.

In particular if $\mathcal{A} = \frac{1}{2}\Delta + L_Z$ and ϕ is closed:

$$\frac{\partial(\delta P_t \phi)}{\partial t}|_{t=0} = \left(\frac{1}{2}\Delta^1 + L_Z\right)\phi.$$

If the s.d.s. considered is a gradient system, we do not require ϕ to be closed in the above.

Proof: Take $v_0 \in T_{x_0}M$. Applying Itô formula to ϕ we have:

$$\lim_{t\to 0} \frac{\delta P_t \phi - \phi}{t}(v_0) = \lim_{t\to 0} \frac{1}{t} \int_0^t E\left(\mathcal{L}\phi_{x_s}(v_s)\chi_{t<\xi}\right) ds = \mathcal{L}\phi_{x_0}(v_0)$$

since $\mathcal{L}$ is a local operator(so $\mathcal{L}\phi$ remain bounded and continuous) and $E|T_{x_0}F_s(v_0)|$ is continuous in s.

6.4 Analysis of δP_t for Brownian systems

This section will be devoted to discussions of δP_t in the special case of $\mathcal{A} = \frac{1}{2}\triangle^h$. The situation here is particular nice, since the generator are both self-adjoint and elliptic. However to start with we would like to mention the following relevant theorem for a general elliptic generator, which is an improvement of a theorem of Elworthy from [26](part 1,3 and 4 are new):

As usual let $\{U_n\}$ be a sequence of nested relatively compact open sets in M with $\bar{U}_n \subset U_{n+1}$ and $\cup U_n = M$. Denote by T_n the first exit time of $F_t(x)$ from U_n.

Theorem 6.4.1 *Let $\mathcal{A} = \frac{1}{2}\triangle + L_Z$. Let $\{\phi_t : t > 0\}$ be a regular solution of the heat equation:*

$$\frac{\partial}{\partial t}\phi_t = \frac{1}{2}\triangle^1\phi_t + L_Z\phi_t$$

with $d\phi_t = 0$ for all t. Then $\phi_t(v_0) = (\delta P_t)\phi_0(v_0)$ if one of the following conditions hold:

1. *Suppose $|\phi_s|$ and $|\nabla\phi_s|$ are uniformly bounded in s in $[0,t]$ for any t, and assume the s.d.s. is complete with $|\nabla X|$ bounded and*

$$\int_0^t E|T_xF_s|^2 ds < \infty, \quad x \in M.$$

2. *Assume completeness, ϕ_0 bounded, and $E\left(\sup_{s\leq t}|T_xF_s|\right) < \infty$ for each t and x.*

3. *Suppose $|\phi_s|$ is uniformly bounded in s in $[0,t]$ for all t, and assume the s.d.s. is complete with $\sup_n E\left(|TF_{T_n(x)}|^{1+\delta}\chi_{T_n<t}\right) < \infty$ for some constant $\delta > 0$ and each t.*

4. *Assume for each t, $|\phi_s|$ converges uniformly in $s \in [0,t]$ to zero as $x \to \infty$ and*

$$\overline{\lim_n} E|TF_{T_n(x)}|\chi_{T_n<t} < \infty$$

Proof: (a). Let $x_0 \in M$, and $v_0 \in T_{x_0}M$. First assume completeness. Apply Itô formula to $\phi_{T-t}(x)$ for fixed number $T > 0$, to get:

$$\phi_{T-t}(v_t) = \phi_T(v_0) + \int_0^t \nabla\phi_{T-s}(X(x_s)dB_s)(v_s) + \int_0^t \phi_{T-s}(\nabla X(v_s)dB_s).$$

Let $t = T$, we get:

$$\begin{aligned}\phi_0(v_T) &= \phi_T(v_0) + \int_0^T \nabla\phi_{T-s}(X(x_s)dB_s)(v_s)\\ &\quad + \int_0^T \phi_{T-s}(\nabla X(v_s)dB_s).\end{aligned}$$

Take expectations on both sides to get:

$$\phi_T(v_0) = E\phi_0(v_T)$$

under either of the first two conditions.

(b). For the rest we apply the Itô formula with stopping time:

$$\begin{aligned}\phi_{T-t\wedge T_n}(v_{t\wedge T_n}) &= \phi_T(v_0) + \int_0^{t\wedge T_n} \nabla\phi_{T-s}(X(x_s)dB_s)(v_s)\\ &\quad + \int_0^{t\wedge T_n} \phi_{T-s}(\nabla X(v_s)dB_s).\end{aligned}$$

Setting $t = T$, we get:

$$\phi_0(v_T)\chi_{T<T_n} + \phi_{T-T_n}(v_{T_n})\chi_{T>T_n}$$

$$= \phi_T(v_0) + \int_0^{T\wedge T_n} \nabla\phi_{T-s}(X(x_s)dB_s)(v_s)$$
$$+ \int_0^{T\wedge T_n} \phi_{T-s}(\nabla X(v_s)dB_s).$$

Taking expectations of both sides above to get:

$$E\left(\phi_0(v_T)\chi_{T<T_n}\right) + E\left(\phi_{T-T_n}(v_{T_n})\chi_{T>T_n}\right) = \phi_T(v_0).$$

(c). Assuming part 3, $\phi_{T-T_n}(v_{T_n})\chi_{T>T_n}$ is uniformly integrable. So

$$\lim_{n\to\infty} E\left(\phi_{T-T_n}(v_{T_n})\chi_{T>T_n}\right) = E\left(\lim_{n\to\infty} \phi_{T-T_n}(v_{T_n})|\chi_{T>T_n}\right) = 0.$$

since $T_n \to \infty$ from completeness while

$$E\phi_0(v_T)\chi_{T<T_n} \to \delta P_t(\phi_0)(v_0).$$

(d). For part 4: Let $\epsilon > 0$. There is a number $N > 0$ s.t. if $n > N$,

$$|\phi_t(x)| < \epsilon \text{ for all } t \text{ and } x \notin U_n \,.$$

Thus

$$E\left(|\phi_{T-T_n}(v_{T_n})|\chi_{T>T_n}\right) \leq \epsilon E\left(|v_{T_n}|\chi_{T>T_n}\right).$$

Letting $\epsilon \to 0$, we get $\overline{\lim}_n E|\phi_{T-T_n}(v_{T_n})|\chi_{T>T_n}| = 0$. Thus the result. ■

Remarks: (1). For gradient system, we do not need the assumption $d\phi_s = 0$ in the theorem.

(2).Let M be a complete Riemannian manifold and $\mathcal{A} = \frac{1}{2}\Delta^h$. This theorem gives us the equivalence of δP_t and $e^{\frac{1}{2}t\Delta^{h,1}}\phi$ for closed forms, while corollary 6.3.5 only gives us the equivalence for exact forms.

(3). Let $\phi \in C_K^\infty$. Then $e^{\frac{1}{2}t\Delta^{h,1}}\phi$ is a regular solution to the heat equation. But we know $de^{\frac{1}{2}t\Delta^{h,1}}\phi = e^{\frac{1}{2}t\Delta^h}d\phi$ from section 2.3. So in the above theorem we only need $d\phi_0 = 0$ instead of $d\phi_t = 0$ for all t.

Corollary 6.4.2 *Let M be a manifold with Ric-2Hess(h) bounded from below. Consider a Brownian system with gradient drift. If for some constant $\delta > 0$:*

$$\sup_n E|TF_{T_n}|^{1+\delta}\chi_{T_n<t} < \infty,$$

then $e^{\frac{1}{2}t\Delta^h}\phi = E\phi(TF_t)$ for bounded closed 1-forms ϕ. Note the required inequality holds if $H_{1+\delta}(v,v) \leq c|v|^2$ for some constsnt c.

Proof: Note there is no explosion by [5]. Furthermore if Ricci-2Hess(h) is bounded from below by a constant c, then

$$|W_t^h| \leq e^{ct}$$

as in [25]. Here W_t^h is the Hessian flow satisfying:

$$\frac{DW_t^h}{\partial t} = -\frac{1}{2}\mathrm{Ric}(W_t^h, -)^{\#} + < \mathrm{Hess}(h)(W_t^h), - >^{\#}.$$

Therefore $e^{\frac{1}{2}t\Delta^h}\phi = E\phi(W_t^h)$ for bounded 1-form ϕ. See [26] for a proof. Hence $e^{\frac{1}{2}t\Delta^h}\phi$ is uniformly bounded in t in finite intervals. Apply part 3 of theorem 6.4.1 and lemma 5.3.4, we get the conclusion. End of the proof. ■

Note also if $h = 0$(use W_t for W_t^h), we may get more information on the heat semigroup:

$$\begin{aligned} |e^{\frac{1}{2}t\Delta^h}\phi| &\leq E|\phi|^2 E|W_t|^2 \\ &\leq e^{2ct}P_t(|\phi|^2). \end{aligned}$$

But P_t has the C_0 property if Ricci is bounded from below. Thus $e^{\frac{1}{2}t\Delta^h}\phi(x)$ converges to zero uniformly in t in finite intervals as x goes to infinity.

For more discussions on ralations between the Ricci flow and the derivative flow, see [29].

The following result does not assume completeness. It is interesting since it does not assume any condition on the heat semigroup either. In fact this is used in the next chapter to get a nonexplosion result. See proposition 7.2.2.

Proposition 6.4.3 *Let $p > 1$, take q to be the conjugate number to p: $\frac{1}{p} + \frac{1}{q} = 1$. Then δP_t is a L^p semigroup, if for each $t > 0$:*

$$\sup_M E\left(|T_x F_t|^q \chi_{t<\xi}\right) < \infty. \tag{6.5}$$

Also if there is a number $a > 0$ such that:

$$\sup_{s \le a} \sup_M E\left(|T_x F_s|^q \chi_{s<\xi}\right) = c < \infty. \tag{6.6}$$

and the map $t \mapsto |T_x F_t| \chi_{t<\xi}$ is continuous into $L^p(\Omega, \mathcal{F}, P)$ at $t = 0$ for each x, then δP_t is a strongly continuous L^p semigroup.

Furthermore if also $\lim_{t\to 0} \sup_{x\in M} E\left(|T_x F_t| \chi_{t<\xi}\right) = 1$, then δP_t is a strongly continuous semigroup in L^r for $p \le r \le \infty$.

Proof: Take a 1-form $\phi \in L^p$. Then

$$\begin{aligned}
\int |\delta P_t \phi|^p e^h dx &\le \int E\left(|\phi|^p_{F_t(x)} \chi_{t<\xi}\right) \left(E|T_x F_t|^q \chi_{t<\xi}\right)^{\frac{p}{q}} e^h dx \\
&\le \left(\sup_x E|T_x F_t|^q \chi_{t<\xi}\right)^{p-1} \int E\left(|\phi|^p_{F_t(x)} \chi_{t<\xi}\right) e^h dx \\
&= \left(\sup_x E|T_x F_t|^q \chi_{t<\xi}\right)^{p-1} \int |\phi|^p_x e^h dx.
\end{aligned}$$

The last equality comes from the fact that $e^h dx$ is the invariant measure for F_t. So we have showed $\delta P_t \phi$ is in L^p.

To show the strong continuity of δP_t in t, we only need to prove $\delta P_t \phi$ is continuous for ϕ in C_K^∞ by the uniform boundedness principle(see P_{80} in [22]).

Take $\phi \in C_K^\infty$. First we have pointwise continuity from the L^p continuity of $|TF_t| \chi_{t<\xi}$:

$$\lim_{t\to 0} |E\phi(T_x F_t(v_0)) \chi_{t<\xi} - \phi(x)(v_0))|^p = 0$$

for all x in M and $v_0 \in T_xM$.

Next we show $\delta P_t\phi$ converges to ϕ in $L^p(M, e^h dx)$. Let $t < a$, then

$$\begin{aligned} |E(\phi(T_xF_t)\chi_{t<\xi})|^p &\le E\left(|\phi|^p_{F_t(x)}\chi_{t<\xi}\right) \sup_{t\le a}\sup_{x\in M}\left(E\left(|T_xF_t|^q\chi_{t<\xi}\right)\right)^{\frac{p}{q}} \\ &= c^{\frac{p}{q}} E\left(|\phi|^p_{F_t(x)}\chi_{t<\xi}\right). \end{aligned}$$

By the invariance property, we also have:

$$\int_M E\left(|\phi|^p_{F_t(x)}\chi_{t<\xi}\right) e^h dx = \int_M |\phi|^p_x e^h dx.$$

But

$$\begin{aligned} |\delta P_t\phi - \phi|^p &\le c_p|\delta P_t\phi|^p + c_p|\phi|^p \\ &\le c^{\frac{p}{q}} c_p E\left(|\phi|^p_{F_t(x)}\chi_{t<\xi}\right) + c_p|\phi|^p. \end{aligned}$$

Here c_p is a constant. For the last term we may change order of taking expectation and taking limit in t. Thus by a standard comparison theorem, we have:

$$\lim_{t\to 0}\int_M |\delta P_t\phi - \phi|^p e^h\, dx = \int_M \lim_{t\to 0}|E\phi(v_t)\chi_{t<\xi} - \phi|^p e^h dx.$$

So $\delta P_t\phi$ converges to ϕ in L^p.

Finally notice with the last assumption, we can prove that δP_t is a strongly continuous semigroup both on L^p and L^∞. It is a L^∞ semigroup since for $\phi \in L^\infty$,

$$|\delta P_t\phi|_{L^\infty} \le \sup_{x\in M} E|T_xF_t\chi_{t<\xi}||\phi|_{L^\infty}.$$

Its strong continuity comes from the last assumption.

By the Riesz-Thorin interpolation theorem δP_t is a semigroup on L^r for $p \le r \le \infty$. Furthermore:

$$\lim_{t\to 0}\int_M |\delta P_t\phi - \phi|^r e^h dx \le \lim_{t\to 0}\int |\delta P_t\phi - \phi|^p\, |\delta P_t\phi - \phi|^{r-p}_{L^\infty} e^h dx = 0.$$

End of the Proof. ∎

Remark: This proof works whenever there is an invariant measure for F_t.

Corollary 6.4.4 *For a gradient Brownian system with drift ∇h, we have*

$$\delta P_t \phi = e^{\frac{1}{2}t\Delta^{h,1}} \phi$$

for $\phi \in C_K^\infty$ if for all t and some constant $a > 0$

$$\sup_M E|T_x F_t|^2 \chi_{t<\xi} < \infty,$$

$$\sup_{s\leq a} \sup_M E|T_x F_t|^2 \chi_{t<\xi} < \infty,$$

and the map $t \mapsto |T_x F_t| \chi_{t<\xi}$ is continuous into $L^2(\Omega, \mathcal{F}, P)$.

In particular the conditions hold if there is a number $\delta > 0$ such that for all t:

$$\sup_{s\leq t} \sup_M E|T_x F_t|^{2+\delta} \chi_{t<\xi(x)} < \infty. \tag{6.7}$$

Proof: From proposition 6.3.6, the semigroup δP_t has generator $\frac{1}{2}\Delta^h$ for gradient system on C_K^∞. The result follows from theorem 1.4.1 on page 18. ∎

Recall we defined H_p for gradient h-Brownian system (section 5.3) as follows:

$$\begin{aligned} H_p(v,v) = \quad & -\mathrm{Ric}(v,v) + 2 < \mathrm{Hess}(h)(v), v > \\ & + |\alpha(v,\cdot)|^2_{H.S} + \tfrac{p-2}{|v|^2}|\alpha(v,v)|^2. \end{aligned}$$

Following Strichartz, we discuss the L^p boundedness of heat semigroups for forms:

Corollary 6.4.5 *Let M be a complete Riemannian manifold with Ricci-2Hess(h) bounded from below. Suppose there are constants k and $\delta > 0$ such that*

$$H_{1+\delta}(v,v) \leq k_1|v|^2.$$

Then $e^{\frac{1}{2}t\Delta^h}$ is L^p bounded uniformly in t in finite intervals $[0,T]$, for all p between $\frac{1+\delta}{\delta}$ and infinity.

Proof: Consider a gradient Brownian system with generator $\frac{1}{2}\Delta^h$. It has no explosion if Ricci-2Hess(h) is bounded from below [5]. Let $\phi \in C_K^\infty$, corollary 6.4.2 on page 99 gives us:

$$e^{\frac{1}{2}t\Delta^h}\phi = E\phi(v_t).$$

Let $\alpha = \frac{1+\delta}{\delta}$ be the conjugate number to $1+\delta$. We have:

$$\begin{aligned}\int_M |E\phi(TF_t)|^\alpha e^h dx &\le \int_M E|\phi|^\alpha (E|TF_t|^{1+\delta})^{\frac{1}{\delta}} e^h dx\\ &\le \sup_{t\le T}\sup_x \left(E|T_xF_t|^{1+\delta}\right)^{\frac{1}{\delta}} (|\phi|_{L^\alpha})^\alpha .\end{aligned}$$

However $\sup_{t\le T}\sup_M E|T_xF_t|^{1+\delta} < \infty$ if $H_{1+\delta}(v,v) \le k|v|^2$ as in section 5.3. Thus the L^p boundedness. ∎

Next we consider the L^p contractivity of heat semigroups (see page 18 for definition). Define $\delta P_t\phi$ for k forms as follows:

$$\delta P_t\phi(v_0^1,\ldots,v_0^k) = E\phi(TF_t(v_0^1),\ldots,TF_t(v_0^k))\chi_{t<\xi}. \tag{6.8}$$

Lemma 6.4.6 *Consider a Brownian system with drift ∇h on a complete Riemannian manifold. Suppose $\delta P_t\phi = e^{\frac{1}{2}t\Delta^h}\phi$ for k forms $\phi \in L^\infty$. If there is a constant $\delta > 0$ such that*

$$\sup_x E|T_xF_t\chi_{t<\xi(x)}|^{k(1+\delta)} \le 1,$$

Then the heatsemigroup $e^{\frac{1}{2}t\Delta^h}$ on k forms is contractive on L^p (on $L^2\cap L^p$), for p between $\frac{1+\delta}{\delta}$ and ∞.

Proof: Let $\phi \in C_K^\infty$. By the argument in the proof of the last corollary, we have

$$|e^{\frac{1}{2}t\Delta^h}\phi|_{L^{\frac{1+\delta}{\delta}}} \le |\phi|_{L^{\frac{1+\delta}{\delta}}}.$$

On the other hand we have, for $\phi \in L^\infty$:

$$|e^{\frac{1}{2}t\Delta^h}\phi|_{L^\infty} \le |\phi|_{L^\infty}.$$

Thus $e^{\frac{1}{2}t\Delta^h}$ is both $L^{\frac{1+\delta}{\delta}}$ and L^∞ contractive by the uniform boundedness principle. Finally we apply the Reisz-Thorin interpolation theorem to get the required result. ∎

Let M be a compact manifold. Then $\delta P_t\phi = e^{\frac{1}{2}t\Delta^{h,1}}\phi$ for closed C^2 1-form ϕ, and $\delta P_t = e^{\frac{1}{2}t\Delta^h}$ for C^2 forms of all order if we are considering gradient systems. See [26] for detail. From this we have:

Corollary 6.4.7 *Let M be a compact manifold. Consider a gradient Brownian system with generator $\frac{1}{2}\Delta$. Then*

(1). The first real cohomology group vanishes if the Ricci curvature is positive definite at one point and there is a number $\delta > 0$ such that

$$\sup_M E|T_xF_t|^{1+\delta} \le 1$$

when t large.

(2). The cohomology in dimension k vanishes for a manifold whose curvature operator is positive definite at one point or for a 2-dimensional manifold which is not flat if our gradient Brownian system satisfies

$$\sup_M E|T_xF_t|^{k(1+\delta)} \le 1.$$

when t large.

Proof: We apply the following theorem from [63]: Suppose the real cohomology in dimension k is not trivial, then the heat semigroup for k forms is not L^p contractive for $p \neq 2$ in the following cases:

(1). $n = 2$ and the manifold is not flat.

(2). $k = 1$ and the Ricci curvature is strictly positive at one point.

(3). The curvature operator is positive definite at one point.

Note we have L^p contractivity for some $p \neq 2$ from the assumption, thus finishing the proof. ■

Example:

Let $M = S^n(r)$ be the n-sphere of radius r in R^{n+1}. Then the second fundamental form α is given by: $\alpha_x(u, v) = -\frac{1}{r^2} < u, v > x$. The derivative flow for the gradient Brownian motion on it satisfy:

$$E|v_t|^p = |v_0|^p + \frac{p}{2} \int_0^t E|v_s|^{p-2} H_p(v_s, v_s) ds,$$

and $H_p(v, v) = (p - n)\frac{|v|^2}{r^2}$. So

$$E|v_t|^p = |v_0|^p e^{\frac{p(p-n)}{2|r|^2} t}.$$

See Elworthy [26]. Thus

$$\sup_M E|T_x F_t|^p \leq e^{\frac{p(p-n)}{2|r|^2} t},$$

which is less or equal to 1, when t big if $p \leq n$. The above corollary confirms that the kth cohomology vanishes for the n-sphere if $k < n$. However note that the n^{th} cohomoloy of the sphere does not vanish.

We'll come back to this topic in the next chapter.

Chapter 7

Consequences of moment stability

7.1 Introduction

In the first section, we will show the first homotopy group vanishes if the Brownian motion on M is strongly moment stable and satisfies certain regularity conditions, as for compact manifolds in [27]. Also interesting here is the result on finite h-volume of the manifold. In section 2, we look at the existence of harmonic functions and at cohomology given strong moment stability.

7.2 Geometric consequences and vanishing of $\pi_1(M)$

We shall first show if the heat semigroup for 1-forms is "continuous" on L^∞, then the h-Brownian motion on M has no explosion. In the following lemma we only used pointwise conditions, improving a result of Bakry [5] proved by essentially the same method.

Lemma 7.2.1 *Let M be a complete Riemannian manifold. Assume there is a point $x_0 \in M$ with a neighbourhood U such that for each $x \in U$ there is a constant $C_t(x)$ with the following property:*

$$|e^{\frac{1}{2}t\Delta^{h,1}} df|_x \leq C_t(x)|df|_\infty,$$

for $f \in C_K^\infty$ and $t \leq t_0$. Here t_0 is a positive constant. Then the h-Brownian motion does not explode.

Proof: Let h_n be an increasing sequence in C_K^∞ such that $\lim_{n\to\infty} h_n \to 1$, $0 \leq h_n \leq 1$, and $|\nabla h_n| < \frac{1}{n}$. Such a sequence exists as shown in the appendix. Then $e^{\frac{1}{2}t\Delta^h} h_n(x) \to e^{\frac{1}{2}t\Delta^h} 1(x)$ for each x, since $e^{\frac{1}{2}t\Delta^h} h_n = P_t h_n$ and by the bounded convergence theorem. Here $e^{\frac{1}{2}t\Delta^h} 1$ is defined as in theorem 1.6.1. By Schauder type estimate as in the appendix, we have:

$$de^{\frac{1}{2}t\Delta^h} h_n(x) \to d(e^{\frac{1}{2}t\Delta^h} 1)(x)$$

for all x in M. However for $t \leq t_0$, and $x \in U$

$$\begin{aligned} |de^{\frac{1}{2}t\Delta^h} h_n(x)| &= |e^{\frac{1}{2}t\Delta^{h,1}}(dh_n)|_x \\ &\leq C_t(x)|dh_n|_\infty \leq \frac{C_t(x)}{n} \to 0. \end{aligned}$$

Thus $d(e^{\frac{1}{2}t\Delta^h} 1)(x) \equiv 0$ around x_0. So

$$\frac{\partial}{\partial t}(e^{\frac{1}{2}t\Delta^h} 1)(x_0) = \frac{1}{2}\Delta^h(e^{\frac{1}{2}t\Delta^h} 1)(x_0) = 0.$$

This gives: $e^{\frac{1}{2}t\Delta^h} 1(x_0) = 1$, for $t < t_0$. But $P_t 1(x_0) = e^{\frac{1}{2}t\Delta^h} 1(x_0)$ from proposition 6.2.1. Thus $P_{t_0} 1(x_0) = P\{t_0 < \xi(x_0)\} = 1$. Consequently $P_t 1 \equiv 1$ for all t. Next we notice Brownian motion does not explode if it does not explode at one point. The proof is finished. ■

Proposition 7.2.2 *Consider a h-Brownian motion on a complete Riemannian manifold. Assume there is a point $x_0 \in M$ and a neighbourhood U of x_0 such that: $E\left(|T_xF_t|\chi_{t<\xi}\right) < \infty$ for all $x \in U$ and $t < t_0$. Here $t_0 > 0$ is a constant. Then $P_t 1 = 1$ if $dP_t f = (\delta P_t)(df)$ for $f \in C_K^\infty$.*

Proof: This is a direct consequence of lemma 6.3.4 and the lemma above. ■

Note: If $dP_t f = (\delta P_t)(df)$ for f with both f and df bounded, then set $f \equiv 1$, the same argument in lemma 7.2.1 shows $P_t 1 = 1$.

The proposition above indicates there is not much point to discuss the possibility of $dP_t = (\delta P_t)d$ when there is explosion. From this there also arises an interesting question, which we have not answered yet: If $E\left(|TF_t|\chi_{t<\xi}\right) < \infty$ for $t < t_0$, does it hold for all t?

Corollary 7.2.3 *Let (X, A) be a gradient s.d.s. with generator $\frac{1}{2}\triangle^h$. Then it has no explosion if its derivative flow TF_t satisfies the following conditions:*

1. *For each $t > 0$,*
$$\sup_M E|T_xF_t|^2\chi_{t<\xi(x)} < \infty$$
2. *There is a number $a > 0$ such that:*
$$\sup_{s\le a}\sup_{x\in M}\left(E|T_xF_s|^2\chi_{s<\xi(x)}\right) < \infty,$$
3. *The map $t \to |T_xF_t|\chi_{t<\xi(x)}$ is continuous into $L^2(\Omega, \mathcal{F}, P)$.*

Proof: This comes from proposition 7.2.2 and corollary 6.4.4. ■

Note that the conditions in the corollary can be checked in terms of the extrinsic curvatures of the manifold (see section 5.3). Thus this gives us a result on nonexplosion of the Brownian motion(not necessarily gradient) on

M. However it is not clear that this improves Bakry's result: there is no explosion if Ricci-Hess(h) is bounded from below.

Following Bakry [5], we get a finite volume result:

Theorem 7.2.4 *Let M be a complete Riemannian manifold. Assume the s.d.s. has generator $\frac{1}{2}\triangle^h$ and for each $t \geq 0$ and each compact set K we have:*

(1). $dP_t f = (\delta P_t)(df)$, for $f \in C_K^\infty$.

(2).

$$\int_0^\infty \sup_{x\in K} E|T_x F_s \chi_{s<\xi}| ds < \infty.$$

Then the h-volume of the manifold is finite.

Proof: Let $f \in C_K^\infty$ with nonempty support K. Then $\lim_{t\to\infty} P_t f$ exists in L^2 and is h-harmonic: $\triangle^h(\lim_{t\to\infty} P_t f) = 0$. This comes from the self-adjointness of $\triangle^h$ and an application of the spectral theorem. Let $P_\infty f$ be the limit, then $\nabla(P_\infty f) = 0$. So $P_\infty f$ must be a constant.

Assume h-vol$(M) = \infty$, then $P_\infty f$ must be zero. We will prove this is impossible. Take $g \in C_K^\infty$, then:

$$\int_M (P_\infty f - f) g e^h\, dx = \lim_{t\to\infty} \int_M (P_t f - f) g e^h\, dx.$$

But

$$\begin{aligned}
&\int_M (P_t f - f) g e^h dx = \int_M (\int_0^t \frac{\partial}{\partial s} P_s f\, ds) g\, e^h dx \\
&= \frac{1}{2}\int_0^t \left(\int_M (\triangle^h P_s f) g e^h dx\right) ds = \frac{1}{2}\int_0^t \int_M < d(P_s f), dg > e^h dx ds \\
&= \frac{1}{2}\int_0^t \int_M < P_s^{h,1}(df), dg > e^h dx ds = \frac{1}{2}\int_0^t \int_M < df, P_s^{h,1}(dg) > e^h dx ds \\
&= \frac{1}{2}\int_0^t \int_K < df, P_s^{h,1}(dg) > e^h dx\, ds
\end{aligned}$$

since $\triangle^{h,1}$ is self-adjoint and so the dual semigroup $\left(P_t^{h,1}\right)*$ equals $P_t^{h,1}$.

Next note the stochastic dynamical system is complete under the assumption from proposition 7.2.2. Let

$$C_s = \sup_{x \in K} E\left(|T_x F_s|\right).$$

Thus

$$\begin{aligned}
\left|\int_M (P_t f - f) g e^h dx\right| &\leq \frac{1}{2}\int_0^t \int_K |\nabla f|\, E\left(|T_x F_s|\right)\, |\nabla g|_\infty e^h dx\, ds \\
&\leq \frac{1}{2}|\nabla g|_\infty \left(\int_0^t C_s ds\right) \int_K |\nabla f| e^h dx \\
&\leq \frac{1}{2}|\nabla g|_\infty\, |\nabla f|_{L^1} \int_0^t C_s ds.
\end{aligned}$$

Let $g = h_n$, we get:

$$\left|\int_M (P_t f - f) h_n\, e^h dx\right| \leq \frac{1}{2n}|\nabla f|_{L^1} \int_0^t C_s ds \to 0.$$

But $\lim_{n\to\infty} \int_M (-f h_n) e^h dx = -\int_M f e^h dx$, from $|f h_n| \leq |f| \in L^1$. And we can choose a function $f \in C_K^\infty$ with $\int_M f e^h dx \neq 0$. This gives a contradiction. Thus the h-volume of M must be finite. ∎

Corollary 7.2.5 *Let M be a complete Riemannian manifold. Assume the generator is $\frac{1}{2}\Delta^h$ and there is no explosion. Then if for each x and t:*

$$E \sup_{s \leq t} |T_x F_s| < \infty,$$

and for each compact set K

$$\int_0^\infty \sup_{x \in K} E|T_x F_s| ds < \infty,$$

the manifold has finite h-volume.

Proof: By theorem 6.4.1, $dP_t f = \delta P_t f$ under the assumption. Applying the above theorem, we get the conclusion. ∎

Corollary 7.2.6 *Let M be a non-compact complete Riemannian manifold with*

$$\mathrm{Ric}_x > -\frac{n}{n-1}\frac{1}{\rho(x)^2}$$

for each $x \in M$. Then if $d(P_t f) = (\delta P_t)(df)$, (for F_t a Brownian motion on M), it cannot be strongly moment stable(defined on page 14). Here ρ denotes the distance function on M from a fixed point O.

Proof: This is a direct application of the following theorem from [13]: The volume of M is infinite for noncompact manifolds with the above condition on the Ricci curvature. But by theorem 7.2.4 strong moment stability implies finite volume. Note the Brownian motion here has no explosion by the example on page 57. ■

7.3 Vanishing of $\pi_1(M)$

Theorem 7.3.1 *Let M be a Riemannian manifold with its injectivity radius bigger than a positive number c. Assume we have an s.d.s. (X, A) on M which is strongly 1-complete(with C^2 coefficients), then the first homotopy group $\pi_1(M)$ vanishes if for each compact set K,*

$$\lim_{t\to\infty} \sup_{x\in K} E|T_x F_t| = 0.$$

Proof: Take σ to be a C^1 loop parametrized by arc length. Then $F_t \circ \sigma$ is a C^1 loop homotopic to σ by the strong 1-completeness. Let $\ell(\sigma_t)$ denote the length of $F_t(\sigma)$.

If we can show $F_t \circ \sigma$ is contractible to a point in M with probability bigger than zero for some $t > 0$, then the theorem is proved from the definition: $\pi_1(M) = 0$ if every continuous loop is contractible to one point.

We have:

$$\begin{aligned} E\ell(\sigma_t) &= E\int_0^{\ell_0} |T_{\sigma(s)}F_t(\dot\sigma(s))|ds \\ &\leq \ell_0 \sup_s E|T_{\sigma(s)}F_t|. \end{aligned}$$

Thus

$$\lim_{t\to\infty} E\ell(\sigma_t) = 0.$$

Take t_0 such that $E\ell(\sigma_{t_0}) < \frac{c}{2}$. Then $\ell(\sigma_{t_0}) < \frac{c}{2}$ with probability bigger than zero. For such ω wth $\ell(\sigma_{t_0})(\omega) < \frac{c}{2}$, $F_{t_0} \circ \sigma(\omega)$ is contained in a geodesic ball with radius smaller than $\frac{1}{2}c$. Since the geodesic ball is diffeomorphic to a ball in R^n, it is contractible to one point. Thus $F_{t_0} \cdot \sigma(\omega)$ is contractible to one point for a set of ω with probability bigger than 0. This finished the proof. ■

Theorem 7.3.2 *Let M be a complete Riemannian manifold. Consider a s.d.s. (X, A) with generator $\frac{1}{2}\Delta^h$. Suppose the s.d.s. is strongly 1-complete and satisfies $dP_t f = (\delta P_t)(df)$ for $f \in C_K^\infty$. Then the first homotopy group $\pi_1(M)$ vanishes if*

$$\int_0^\infty \sup_{x\in K} E|T_x F_t| dt < \infty.$$

Proof: Let $\sigma : [0, \ell_0] \to M$ be a C^1 loop parametrized by arc length. Then $F_t \circ \sigma$ is again a C^1 loop homotopic to σ from the strong 1-completeness. Denote by $\ell(\sigma_t)$ its length.

We only need to show $F_t\sigma(\omega)$ is contractible to one point for some ω and for some t.

First we claim there is a sequence of numbers $\{t_j\}$ converging to infinity such that:

$$E\ell(\sigma_{t_j}) \to 0. \tag{7.1}$$

Since:

$$\begin{aligned}
\int_0^\infty E\ell(\sigma_t)dt &= \int_0^\infty E\int_0^{\ell_0} |T_{\sigma(s)}F_t|dsdt \\
&= \int_0^\infty \int_0^{\ell_0} E|T_{\sigma(s)}F_t|dsdt \\
&\le \ell_0 \int_0^\infty \sup_s E|T_{\sigma(s)}F_t|dt < \infty.
\end{aligned}$$

So $\underline{\lim}_{t\to\infty} E\ell(\sigma_t) = 0$, giving (7.1). Therefore $\ell(\sigma_{t_j}) \to 0$ in probability.

Let $m = e^h dx$ be the normalized invariant measure on M for the process. Let K be a compact set in M containing the image set of the loop σ and which has measure $m(K) > 0$. Let $a > 0$ be the infimum over $x \in K$ of the injectivity radius at x.

By (7.1), there is a number N such that for $j > N$,

$$P\{\ell(\sigma_{t_j}) > \frac{1}{2}a\} < \frac{m(K)}{4}.$$

Note h-vol$(M) < \infty$ by theorem 7.2.4, the ergodic theorem(see chapter 3) gives:

$$\lim_{t\to\infty} P\{F_t(x) \in K\} = m(K)$$

for all $x \in M$.

Take a point $\bar{x}$ in the image of the loop σ. There exists a number N_1 such that if $j > N_1$, then:

$$P\{F_{t_j}(\bar{x}) \in K\} > \frac{m(K)}{2}.$$

Thus

$$\begin{aligned}
&P\{\ell(\sigma_{t_j}) < \frac{1}{2}a, F_{t_j}(\bar{x}) \in K\} \\
&= P\{F_{t_j}(\bar{x}) \in K\} - P\{F_{t_j}(\bar{x}) \in K, \ell(\sigma_{t_j}) > \frac{1}{2}a\}
\end{aligned}$$

$$\geq P\{F_{t_j}(\tilde{x}) \in K\} - P\{\ell(\sigma_{t_j}) > \frac{1}{2}a\}$$
$$> \frac{m(K)}{4}.$$

But by the definition of the injectivity radius, there is a coordinate chart containing a geodesic ball of radius $\frac{a}{2}$ around $F_{t_j}(\tilde{x})$. So the whole loop $F_{t_j} \circ \sigma$ is contained in the same chart with probability $> \frac{m(k)}{4}$ (for t big), thus contractible to one point. ■

Corollary 7.3.3 *Let M be a complete Riemannian manifold. Assume non-explosion for the h-BM. If $E\left(\sup_{s\leq t}|T_xF_s|\right) < \infty$ and*

$$\int_0^\infty \sup_{x\in K} E|T_xF_t|dt < \infty$$

for every compact set K, then we have $\pi_1(M) = 0$.

In particular if $E\left(\sup_{s\leq t}|T_xF_s|\right) < \infty$ then F. cannot be strongly moment stable given nonexplosion unless M is simply connected.

Proof: Apply theorem 5.2.4, theorem 6.4.1, and the theorem above.

Corollary 7.3.4 *Let M be a complete Riemannian manifold of finite h-volume. Assume the h-Brownian motion on M is strongly 1-complete and satisfies:*

$$\int_0^\infty \sup_{x\in K} E|T_xF_t|dt < \infty$$

for each compact set K. Then $\pi_1(M) = \{0\}$.

Proof: This comes from the proof of theorem 7.3.2. ■

Here is a corollary which generalizes a result of Elworthy and Rosenberg to noncompact manifolds. See [28].

Corollary 7.3.5 *Let M be a complete Riemannian manifold. Suppose there is a h-Brownian system on M such that $|\nabla X|$ is bounded and $H_1(v,v) < -c^2|v|^2$ for $c \neq 0$. Then we have $\pi_1(M) = \{0\}$. In particular if M is a submanifold of R^n, we have $\pi_1(M) = \{0\}$ if its second fundamental form α is bounded, and $H_1(v,v) < -c^2|v|^2$. Here*

$$\begin{aligned} H_p(v,v) &= -Ric(v,v) + 2 < Hess(h)(v), v > \\ &+ \sum_1^m |\nabla X^i(x)|^2 + (p-2)\sum_1^m \frac{1}{|v|^2} < \nabla X^i(v), v >^2 \end{aligned}$$

Proof: There is no explosion since Ricci-2Hess(h) is bounded from below from the assumptions. On the other hand $H_{1+\delta}$ is bounded above since $|\nabla X|$ is bounded and so $d(P_t f) = (\delta P_t)(df)$ for f in C_K^∞. See corollary 6.3.3. The result follows from theorem 7.3.2 since the s.d.s. is strongly moment stable from $H_1(v,v) \leq -c^2|v|^2$. The second part of the theorem follows from the fact that the sum of the last two terms in the formula for H_p is $|\alpha(v,\cdot)|_{H.S}^2 + (p-2)\frac{1}{|v|^2}|\alpha(v,v)|^2$ for gradient Brownian systems.

7.4 Vanishing of harmonic forms and cohomology

We come back to the discussion on cohomology vanishing of page 104 and aim to extend some of the results in [26] on cohomology vanishing given moment stability to noncompact manifolds. See [33],[26].

Let $C^\infty(\Omega^p)$ be the space of C^∞ smooth p forms on M. A p-form ϕ is closed if $d\phi = 0$, exact if $\phi = d\psi$ for some p-1 form ψ. Here d is the exterior differentiation defined in section 1.5. A *h-harmonic* form is a form with $\triangle^h\phi = 0$. The p^{th} *de Rham cohomology* group $H^p(M, R)$ is defined to be the quotient group of the group of smooth closed p forms by the group of C^∞

exact forms:

$$H^p(M,R) = \frac{\mathrm{Ker}\,(d : C^\infty(\Omega^p) \to C^\infty(\Omega^{p+1}))}{\mathrm{Im}\,(d : C^\infty(\Omega^{p-1}) \to C^\infty(\Omega^p))}.$$

There is also the cohomology group $H^p_K(M,R)$ with compact supports:

$$H^p_K(M,R) = \frac{\mathrm{Ker}\,(d : C^\infty_K(\Omega^p) \to C^\infty_K(\Omega^{p+1}))}{\mathrm{Im}\,(d : C^\infty_K(\Omega^{p-1}) \to C^\infty_K(\Omega^p))}.$$

Let h be a C^∞ smooth function. There is the Hodge decomposition theorem(see page 33):

$$L^2\Omega^p = \overline{\mathrm{Im}(\delta^h)} \bigoplus \overline{\mathrm{Im}(d)} \bigoplus L^2(\mathcal{H}).$$

Let ϕ be a form with $d\phi = 0$ and decomposition: $\phi = \alpha + \beta + H\phi$. Here $\alpha \in \overline{\mathrm{Im}(\delta)}$ and $\beta \in \overline{\mathrm{Im}(d)}$. Then $\alpha = 0$ since $d\alpha = 0$ by $d\beta = 0$ and $d(H\phi) = 0$.

Thus we have the Hodge's theorem: every cohomology class has a unique harmonic representative if Im(d) is closed, in particular: the dimension of $H^p(M,R)$, as a linear space, equals the dimension of the space of h-harmonic L^2 $p-$forms when d has closed range.

The vanishing problem have been studied by conventional methods by e.g. Yau [67]. The problem has been considered in a probabilistic context, see e.g. Vauthier [66], Elworthy and Rosenberg [33],[34]. The idea we are using here is that the probabilistic semigroup δP_t on forms often agrees with the heat semigroup (see chapter 6). Thus the existence of harmonic forms is directly related to the behaviour of diffusion processes and their derivatives. In the following we follow Elworthy and Rosenberg's approach to get vanishing results for harmonic 1-forms. But we use δP_t instead of the standard probabilistic formula obtained from the Weitzenbock formula. However we do not intend to include all the possible results in this thesis, but only demonstrate the idea. A second theorem we give here follows from an approach of Elworthy [27]. This approach uses integration of p-forms along singular p-simplices and fits very well with our definition of strong p-completeness.

Proposition 7.4.1 *Let M be a complete Riemannian manifold, consider a s.d.s. with generator $\frac{1}{2}\triangle^h$.*

(1). Assume $e^{\frac{1}{2}t\triangle^{h,1}}\phi = \delta P_t \phi$ for closed 1-forms $\phi \in L^q$. Let p be the conjugate number to q. Then there are no nonzero L^p h-harmonic 1-forms, if

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in M} E\left(|T_x F_t|\chi_{t<\xi}\right)^q < 0. \tag{7.2}$$

(2). Assume $e^{\frac{1}{2}t\triangle^{h,1}}\phi = \delta P_t \phi$ for closed bounded 1-forms ϕ. Then there are no bounded h-harmonic 1-forms if for each $x \in M$

$$\lim_{t\to\infty} \frac{1}{t} \log E\left(|T_x F_t|\chi_{t<\xi}\right) < 0. \tag{7.3}$$

Proof: (1). We have nonexplosion and finite volume in the first case according to the nonexplosion result on page 108. Let ϕ be a nonzero harmonic p-form in L^p. Then there is a point $x_0 \in M$ with $\phi(x_0) \neq 0$. Thus $\int |\phi|_x^p e^h dx > 0$ by continuity. So

$$\begin{aligned}
0 &= \lim_{t\to\infty} \frac{1}{t} \log \int |\phi|_x^p e^h dx \\
&= \lim_{t\to\infty} \frac{1}{t} \log \int |\delta P_t \phi|_x^p e^h dx \\
&= \lim_{t\to\infty} \frac{1}{t} \log \int |E\phi(T_x F_t)|^p e^h dx \\
&\leq \lim_{t\to\infty} \frac{1}{t} \log \int E|\phi|^p_{F_T(x)} \left(E|T_x F_t|^q\right)^{\frac{p}{q}} e^h dx \\
&\leq \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in M} \left(E|T_x F_t|^q\right)^{\frac{p}{q}} + \lim_{t\to\infty} \frac{1}{t} \log \int E|\phi|^p_{F_t(x)} e^h dx \\
&\leq \lim_{t\to\infty} \frac{p}{q}\frac{1}{t} \log \sup_{x\in M} E|T_x F_t|^q + \lim_{t\to\infty} \frac{1}{t} \log \int |\phi|_x^p e^h dx.
\end{aligned}$$

But $\int |\phi|_x^p e^h dx < \infty$, giving a contradiction.

(2). The proof of the second part is just as before. First note we have nonexplosion. Let ϕ be a closed bounded harmonic 1-form. Let $x_0 \in M$ with $|\phi|_{x_0} \neq 0$. Then:

$$
\begin{aligned}
0 &= \lim_{t\to\infty} \frac{1}{t} \log|\phi(x_0)| = \lim_{t\to\infty} \frac{1}{t} \log|e^{\frac{1}{2}t\Delta^{h,1}}\phi(x_0)| \\
&= \lim_{t\to\infty} \frac{1}{t} |E\phi(T_{x_0}F_t)| \\
&\le \lim_{t\to\infty} \frac{1}{t} \log|\phi|_\infty E\left(|T_{x_0}F_t|\right) \\
&\le \lim_{t\to\infty} \frac{1}{t} \log E|T_{x_0}F_t|.
\end{aligned}
$$

But this is impossible from the assumption. End of the proof. ■

Remarks (1). Let $p < 2$. Assume (7.2). Then $\delta P_t = e^{\frac{1}{2}t\Delta^{h,1}}$ on C_K^∞ implies $\delta P_t = e^{\frac{1}{2}t\Delta^{h,1}}$ on L^p. Since in this case $q = \frac{p}{p-1} > p$, so δP_t is a strongly continuous L^p semigroup from proposition 6.4.3. We may then apply uniform boundedness principle.

(2). If we know that M has finite h-volume to start with, then all bounded harmonic 1-forms vanishes if $E\left(sup_{s\le t}|T_xF_s|\right) < 0$ and if F_t is strongly moment stable from theorem 5.2.4.

Following Elworthy [27] for the compact case, we have the following theorem:

Theorem 7.4.2 *Let M be a Riemannian manifold and assume there is a strongly p-complete s.d.s. with strong p^{th}-moment stability. Then all bounded closed p-forms are exact. In particular the natural map from $H_K^p(M, R)$ to $H^p(M, R)$ is trivial.*

Proof: Let σ be a singular p-simplex, and ϕ a bounded closed p-form. We shall not distinguish a singular simplex map from its image. Denote by $F_t^*\phi$ the pull back of the form ϕ and $(F_t)_*\sigma = F_t \circ \sigma$. Then

$$\int_{(F_t)_*\sigma} \phi = \int_\sigma (F_t)^*\phi$$

by definition. But $(F_t)_*\sigma$ is homotopic to σ from the strong p-completeness. Thus:

$$\int_\sigma \phi = \int_{(F_t)_*\sigma} \phi.$$

This gives:

$$\int_\sigma \phi = \int_\sigma (F_t)^* \phi.$$

Take expectations of both sides to obtain:

$$\begin{aligned} E|\int_\sigma \phi| &= \lim_{t\to\infty} E|\int_\sigma (F_t)^*\phi| \\ &\leq |\phi|_\infty \lim_{t\to\infty} \int_\sigma E|TF_t|^p \\ &\leq |\phi|_\infty \lim_{t\to\infty} \sup_{x\in\sigma} E|TF_t|^p \\ &= 0 \end{aligned}$$

from the strong p^{th} moment stability. Thus $\int_\sigma \phi = 0$, and so ϕ is exact by deRham's theorem. ∎

Corollary 7.4.3 *Let M be a complete Riemannian manifold. Assume there is a complete s.d.s. on M with strong p^{th} moment stability and satisfying $\sup_{x\in K} E\left(\sup_{s\leq t} |T_xF_s|^{p+1}\right) < \infty$ for each compact set K. Then all bounded closed p-forms are exact. In particular Suppose M is a closed submanifold of R^m with its second fundamental form bounded. Then if $H_p(v,v) \leq -c^2|v|^2$ for some constant c, then the conclusion holds. Here H_p is as defined on page 85:*

$$\begin{aligned} H_p(v,v) &= -Ric(v,v) + 2 < Hess(h)(v), v > \\ &\quad + |\alpha(v,-)|^2_{H.S} + (p-2)\frac{1}{|v|^2}|\alpha(v,v)|^2. \end{aligned}$$

Proof: Direct applications of the above theorem and theorem 5.2.6.

7.5 Examples

Example 1 Let $M = R^n$, $h(x) = -|x|^2$. Then h-vol$(R^n) < \infty$. Furthermore we have:

$$H_1(v, v) = 2 < \text{Hess}(h)(v), v >= -2|v|^2.$$

Thus the s.d.s. on R^n:

$$dx_t = dB_t - \nabla h(x_t)dt = dB_t - 2x_t dt$$

is strongly complete and strongly p^{th} moment stable.

Example 2 Let B_t^1, B_t^2 be independent Brownian motions on R^1. Then

$$F_t(x, y) = (x + \int_0^t e^{B_s^2 - \frac{s}{2}} dB_s^1, ye^{B_t^2 - \frac{t}{2}})$$

is a Brownian flow on the hyperbolic space H^2. It is strongly complete and satisfies:

$$E \sup_{s \le t} |T_x F_s| < \infty.$$

So $\delta P_t \phi = e^{\frac{1}{2}t\Delta^h} \phi$ for bounded 1-forms. Thus this Brownian system on H^2 is not strongly moment stable since H^2 has infinite volume(c.f. theorem 7.3.2).

Example 3 Consider the Langevin equation on R^2:

$$dx_t = \gamma dB_t - cx_t dt.$$

Here γ and c are constants. The solution can be written down explicitly:

$$x_t = x_0 e^{-ct} + \gamma \int_0^t e^{-c(t-s)} dB_s.$$

It has Gaussian distribution and generator $\frac{1}{2}\Delta^h$ for $h = -\frac{cx^2}{2}$, and has no explosion. Its derivative flow is given by:

$$TF_t(v) = e^{-ct} v$$

and enjoys the following properties:

1.

$$\sup_x E\left(\sup_{s\le t} |T_xF_s|^2\right) = 1,$$

2. Strong 1-completeness from theorem 5.2.4,

3.

$$d(P_tf) = (\delta P_t)(df)$$

for $f \in C_K^\infty$ from theorem 6.3.1,

4.

$$\int_0^\infty \sup_x E|T_xF_s|ds = \int_0^\infty \mathrm{e}^{-cs}ds = 1,$$

5. The solution process is recurrent and has $e^h dx$ as finite invariant measure, since h-vol$(R^n) < \infty$.

If we consider the same equation on $M = R^2 - \{0\}$ instead of on R^2, then all the properties hold except for the strong 1-completeness as shall be shown below.

Clearly part 1 and part 4 hold. The solution is recurrent on $R^2 - \{0\}$ since it is recurrent on R^2. Furthermore $e^h dx$ is still the invariant measure since $R^2 - \{0\}$ has negligible boundary and from the completeness of the s.d.e. on $R^2 - \{0\}$. With these the conclusion of proposition 3.0.5 certainly holds.

Suppose the process is strongly 1-complete. The ergodic property gives us $\pi_1(M) = \{0\}$ from the proof of the theorem 7.3.2 and part 4 of the properties. This is a contradiction. Thus we do not have the strong 1-completeness.

Finally part 3 holds since on $R^2 - \{0\}$, $(\delta P_t)(df)(v) = Edf(v)$ and $P_tf(x) = Ef(x + B_t)$ as on R^2.

In the following we look at some surfaces whose second fundamental forms are bounded. We will show that the Hyperboloid satisfies our hypothesis which imply $\pi_1(M) = 0$, and Brownian motions on both the torus and the cylinder cannot be strongly moment stable. First we recall the basic theory:

Let M be a surface in R^3 parametrized by $x = x(u,v)$. The unit normal vector μ is given by:

$$\mu = \frac{x_u \times x_v}{|x_u \times x_v|}.$$

There is the shape operator $S\colon T_xM \to T_xM$ given by:

$$S(v) =: -D_v\mu.$$

Here D denotes the differentiation on R^3. The second fundamental form $II(u,v) = -l(u,v)\mu$ is given in terms of the shape operator S:

$$l(u,v) = - < S(u), v > . \tag{7.4}$$

Let

$$\begin{array}{lll} e =< \mu, x_{uu} > & f =< \mu, x_{uv} > & g =< \mu, x_{vv} > \\ E =< x_u, x_u > & F =< x_u, x_v > & G =< x_v, x_v > . \end{array}$$

There is then the Weingarten equation:

$$\begin{aligned} -S(x_u) &= \tfrac{fF-eG}{EG-F^2} x_u + \tfrac{eF-fE}{EG-F^2} x_v \\ -S(x_v) &= \tfrac{gF-fG}{EG-F^2} x_u + \tfrac{fF-gE}{EG-F^2} x_v. \end{aligned}$$

Example 4 [Surface of revolution]

Consider the surface given by:

$$(c_1(s)\cos\theta, c_1(s)\sin\theta, c_2(s)).$$

For this surface:

$$E = [c_1(s)]^2, \ \ F = 0, \ \ G = \left\{[c_1\prime(s)]^2 + [c_2\prime(s)]^2\right\}^2,$$

$$\begin{aligned} e &= - \frac{c_1(s)c_2\prime(s)}{\sqrt{[c_1\prime]^2 + c_2\prime]^2}}, \\ f &= 0, \\ g &= \frac{-c_1\prime(s)c_2\prime\prime(s) + c_1\prime\prime(s)c_2\prime(s)}{\sqrt{[c_1\prime(s)]^2 + [c_2\prime(s)]^2}}. \end{aligned}$$

So

$$S\begin{pmatrix}\frac{\partial f}{\partial\theta}\\ \frac{\partial f}{\partial s}\end{pmatrix}=\begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial\theta}\\ \frac{\partial f}{\partial s}\end{pmatrix}.$$

Here

$$K_1=\frac{e}{E}=\frac{-c_2'(s)}{c_1(s)\sqrt{[c_1']^2+[c_2']^2}},$$

$$K_2=\frac{g}{G}=\frac{-c_1'(s)c_2''(s)+c_1''(s)c_2'(s)}{\sqrt{[c_1'(s)]^2+[c_2'(s)]^2}^{\,3}}.$$

The normal vector is:

$$\left(\frac{c_2'\cos\theta}{\sqrt{[c_1']^2+[c_2']^2}},\frac{c_2'\sin\theta}{\sqrt{[c_1']^2+[c_2']^2}},-\frac{c_1'}{\sqrt{[c_1']^2+[c_2']^2}}\right).$$

Example 4a [The Hyperboloid]

We will show the surface

$$z^2-(x^2+y^2)=1$$

satisfies the conditions of theorem 5.3.6 and theorem 7.3.2. Consider the following parametrization: $(s\cos\theta, s\sin\theta, \sqrt{s^2+1})$.

Thus $E=s^2$, $F=0$, $G=\frac{2s^2+1}{s^2+1}$, $e=-\frac{s^2}{\sqrt{1+2s^2}}$, $f=0$, $g=-\frac{1}{(1+s^2)\sqrt{1+2s^2}}$.

The unit normal vector is:

$$\mu=\left(\frac{s\cos\theta}{\sqrt{1+2s^2}},\frac{s\cos\theta}{\sqrt{1+2s^2}},-\frac{\sqrt{1+s^2}}{\sqrt{1+2s^2}}\right).$$

Also the Ricci curvature is given by: $\mathrm{Ricci}(v)=K_1K_2|v|^2$, while $K_1=-\frac{1}{\sqrt{1+2s^2}}$, $K_2=-\frac{1}{\sqrt{1+2s^2}^3}$. Clearly the second fundamental form is bounded, thus the Brownian motion on the surface is strongly complete. Next we construct a Brownian motion with drift which is strongly moment stable and thus verify $\pi_1(M)=0$.

According to section 5.3:

$$\begin{aligned}
< \nabla X^i(v), v > &= < A_x(v, < e_i, \mu > \mu), v > \\
&= - < e_i, \mu > l(v, v).
\end{aligned}$$

Let $h = -c|x|^2$. Let grad$\bar{h}$ and H$\bar{\text{e}}$ss(h) denote the gradient and the Hessian of h in R^3 respectively. Then for $x = (x_1, x_2, x_3) \in M$,

$$\begin{aligned}
< \mathrm{Hess}(h)(v), v > &= < \nabla X(v)(\mathrm{grad}\bar{h}), v > + < \mathrm{H\bar{e}ss}h(v), v > \\
&= -2c < \nabla X(v)(x), v > -2c|v|^2 \\
&= -2c \sum_1^3 x_i < e_i, \mu > l(v, v) - 2c|v|^2 \\
&= \frac{2c}{\sqrt{1 + 2s^2}} l(v, v) - 2c|v|^2.
\end{aligned}$$

So

$$< \mathrm{Hess}(h)(v), v >= \frac{2c}{\sqrt{1 + 2s^2}} l(v, v) - 2c|v|^2.$$

Since $l(v, v)$ is negative and bounded, we may choose c big such that

$$\begin{aligned}
H_1(v, v) &= -\mathrm{Ric}(v, v) + |l(v, \cdot)|^2 - \frac{l(v, v)^2}{|v|^2} + 2 < \mathrm{Hess}(h)(v), v > \\
&= -K_1 K_2 |v|^2 + |l(v, \cdot)|^2 - \frac{l(v, v)^2}{|v|^2} + \frac{4c}{\sqrt{1 + 2s^2}} l(v, v) - 4c|v|^2 \\
&\leq -|v|^2.
\end{aligned}$$

The system with the chosen drift is then strongly moment stable, so satisfies the conditions of theorem 7.3.2.

Example 4b The torus given by

$$((a + b\cos v)\cos u, (a + b\cos u)\sin u, b\sin u)$$

has: $E = (a + b\cos u)^2$, $F = 0$, $G = b^2$, $e = \cos u(a + b\cos u)$, $f = 0$,

$g = b$, $K_1 = -\frac{\cos u}{a+b\cos u}$, and $K_2 = -\frac{1}{b}$. So the Brownian motion here is strongly complete, but it cannot be strongly moment stable since $\pi_1(M) \neq 0$. In fact for $a = b = \frac{1}{\sqrt{2}}$, the first moment is identically 1 as calculated, e.g. in [26].

Example 4c The cylinder $S^1 \times (-\infty, \infty)$ parametrized by $(\cos\theta, \sin\theta, s)$ has $K_1 = -1$ and $K_2 = 0$. The normal vector is:

$$\mu = (\cos\theta, \sin\theta, 0).$$

As for torus, the Brownian motion here is strongly complete but not strongly moment stable. To convince ourself we will try to add a drift as for the Hyperboloid. Let $h = -c|x|^2$. Then

$$\begin{aligned} < \mathrm{Hess}(h)(v), v > &= -2cI(v,v) - 2c|v|^2 \\ &= 2c|v_1|^2 - 2c|v|^2 = -2c|v_2|^2. \end{aligned}$$

Here $v = (v_1, v_2)$. Now

$$H_1(v,v) = v_2^2\left(\frac{v_1^2}{|v|^2} - 2c\right) = |v_2|^2 \frac{(1-2c)v_1^2 - 2cv_2^2}{v_1^2 + v_2^2}.$$

Clearly the same argument does not work.

Chapter 8

Formulae for the derivatives of the solutions of the heat equations

8.1 Introduction

In chapter 6, we examined carefully δP_t and the heat semigroup $e^{\frac{1}{2}t\Delta^{h,1}}$ for one forms, and obtained some conditions to ensure $\delta P_t = e^{\frac{1}{2}t\Delta^{h,1}}$. In fact we may expect more once we know δP_t does agree with $e^{\frac{1}{2}t\Delta^{h,1}}$. The semigroup $e^{\frac{1}{2}t\Delta^{h,1}}$ can be given in terms of the line integral 8.1 and a martingale, following from Elworthy [26] for compact manifolds. In particular we have a formula for the gradient of the logarithm of the heat kernel, extending Bismut's formula [8]. See [16] for an infinite dimensional version of the formulae by Da Prato, Elworthy, and Zabczyk. See also Norris [57] for another approach.

The discussions for one forms also work well for higher order forms. We define δP_t for q forms as on page 103, and look briefly the relation between δP_t and the heat semigroup $e^{\frac{1}{2}t\Delta^h}$ for q forms. In the end we give a formula for the exterior derivative of $e^{\frac{1}{2}t\Delta^h}\phi$ in terms of the derivative flow of a h-

Brownian motion and ϕ itself.

In the following we write $P_t^h\phi = e^{\frac{1}{2}t\Delta^h}\phi$. If ϕ is a q-form, we may use $P_t^{h,q}\phi$ instead of $P_t^h\phi$.

8.2 For 1-forms

Let ϕ be a 1-form, we define $\int_0^t \phi \circ dx_s$ to be the line integral of ϕ along Brownian paths as in [26]:

$$\int_0^t \phi \circ dx_s = \int_0^t \phi(X(x_s)dB_s) - \frac{1}{2}\int_0^t \delta^h\phi(x_s)\,ds. \tag{8.1}$$

Here is the formula for 1-form, which is a direct extension of the formula in [26] for compact manifolds:

Proposition 8.2.1 *Let (X, A) be a complete C^2 stochastic dynamical system on a complete Riemannian manifold M with generator $\frac{1}{2}\Delta^h$. Suppose for closed 1-form ϕ in $D(\bar{\Delta}^h) \cap L^\infty$,*

$$(\delta P_t)\phi = e^{\frac{1}{2}t\Delta^{h,1}}\phi$$

and for each $x \in M$,

$$\int_0^t E|T_xF_s|^2 ds < \infty.$$

Then

$$P_t^{h,1}\phi(v_0) = \frac{1}{t}E\int_0^t \phi \circ dx_s \int_0^t < X(x_s)dB_s, TF_s(v_0) > \tag{8.2}$$

for all $v_0 \in T_{x_0}M$.

Proof: Following the proof for a compact manifold as in [26]. Let

$$Q_t(\phi) = -\frac{1}{2}\int_0^t P_s^h(\delta^h\phi)ds. \tag{8.3}$$

Differentiate equation 8.3 to get:

$$\frac{\partial}{\partial t}Q_t\phi = -\frac{1}{2}P_t^h(\delta^h\phi).$$

We also have:

$$\begin{aligned} d(Q_t\phi) &= -\frac{1}{2}\int_0^t d\delta^h(P_s^h\phi)ds \\ &= \frac{1}{2}\int_0^t \triangle^h(P_s^h\phi)ds \\ &= P_t^h\phi - \phi \end{aligned}$$

since $d\delta^h(P_s^h\phi) = P_s^h(d\delta^h\phi)$ is uniformly continuous in s and

$$d(P_s^h\phi) = P_s^h d\phi = 0$$

from proposition 2.3.1. Consequently:

$$\triangle^h(Q_t(\phi)) = -P_t^h(\delta^h\phi) + \delta^h\phi.$$

Apply Itô formula to $(t,x) \mapsto Q_{T-t}\phi(x)$, which is smooth because $P_s^h\phi$ is, to get:

$$\begin{aligned} Q_{T-t}\phi(x_t) &= Q_T\phi(x_0) + \int_0^t d(Q_{T-s}\phi)(X(x_s)dB_s) \\ &\quad + \frac{1}{2}\int_0^t \triangle^h Q_{T-s}\phi(x_s)ds + \int_0^t \frac{\partial}{\partial s}Q_{T-s}\phi(x_s)ds \\ &= Q_T\phi(x_0) + \int_0^t P_{T-s}^h(\phi)(X(x_s)dB_s) - \int_0^t \phi \circ dx_s. \end{aligned}$$

Let $t = T$. We obtain:

$$\int_0^T \phi \circ dx_s = Q_T(\phi)(x_0) + \int_0^T P^h_{T-s}(\phi)(X(x_s)dB_s),$$

and thus

$$E \int_0^T \phi \circ dx_s \int_0^T < X(x_s)dB_s, TF_s(v_0) >= E \int_0^T P^h_{T-s}\phi(TF_s(v_0))ds.$$

But

$$E \int_0^T P^h_{T-s}\phi(TF_s(v_0))ds = \int_0^T EP^h_{T-s}\phi(TF_s(v_0))ds. \tag{8.4}$$

Since by Fubini's theorem we only need to show $EP^h_{T-s}\phi(TF_s(v_0))$ is integrable with respect to the double integral:

$$\int_0^T E|P^h_{T-s}\phi(TF_s(v_0))|ds \quad \leq \quad |\phi|_\infty \int_0^T E|TF_T(v_0)|ds < \infty.$$

Next notice:

$$E\left(P^h_{T-s}\phi((TF_s(v_0))\right) = E\phi(TF_T(v_0)) = P^h_T\phi(v_0).$$

from the strong Markov property. We get:

$$P^{h,1}_T\phi(v_0) = \frac{1}{T}E\left\{\int_0^T \phi \circ dx_s \int_0^T < XdB_s, TF_s(v_0) >\right\}.$$

End of the proof. ■

Remark: If we assume $\sup_s E|T_sF_t|^2 < \infty$ for each t in the proposition, we do not need to assume $\phi \in L^\infty$. Since first we have $\delta P_t\phi = e^{\frac{1}{2}t\Delta^h}\phi$ for such ϕ by the uniform boundedness principle and also equation (8.4) holds from the following argument:

$$\int_0^T E|P^h_{T-s}\phi(TF_s(v))|ds$$
$$= \int_0^T E|\{E\{\phi(TF_{s,T})|\mathcal{F}_s\}(TF_s(v))|ds$$
$$\le \int_0^T E|\phi(TF_T(v))|\, ds$$
$$\le \sup_x E|T_xF_T(v)|^2 \left(\int_0^T E|\phi|^2_{F_T(x)}ds\right)$$
$$\le T\ E\left(|\phi|^2_{F_T(x)}\right) \sup_x E|T_xF_T(v)|^2 < \infty.$$

But $\int_M E|\phi|^2_{F_T(x)}e^h dx = \int |\phi|^2 e^h dx < \infty$. So $E|\phi|_{F_T(x)} < \infty$ since $E|\phi|^2_{F_T(x)} = P_T(|\phi|^2)(x)$ is continuous in x. Thus we may still apply Fubini's theorem to get: (8.4).

When $\phi = df$ for some function f, formula 8.2 may be rewritten as:

$$d(P_t^h f)(v_0) = \frac{1}{t}Ef(x_t)\int_0^t < TF_s(v_0), X(x_s)dB_s > . \tag{8.5}$$

In fact this works in a more general situations. Here is a a very intuitive proof by D. Elworthy and myself (let BC^1 be the space of bounded functions with bounded continuous first derivative):

Theorem 8.2.2 *Let (X, A) be a complete nondegenerate stochastic dynamical system so there is a right inverse map $Y(x)$ for $X(x)$ each x in M. Let f be in BC^1 s.t. $(\delta P_t)(df) = d(P_t f)$. Then for $v_0 \in T_{x_0}M$:*

$$d(P_t f)(v_0) = \frac{1}{t}Ef(x_t)\int_0^t < dB_s, Y(TF_s(v_0)) > \tag{8.6}$$

provided $\int_0^t < dB_s, Y(T_xF_s(v_0)) >$ is a martingale for all t. Here $P_t f$ is a solution to $\frac{\partial}{\partial t} = \frac{1}{2}\sum X^i X^i + A$ with initial value f.

Proof: Let $T > 0$. Apply Itô formula to the smooth map $(t, x) \mapsto P_{T-t}f(x)$:

$$P_{T-t}f(x_t) = P_Tf(x_0) + \int_0^t dP_{T-s}f(x_s)(XdB_s).$$

Letting t=T, we have:

$$f(x_T) = P_Tf(x_0) + \int_0^T dP_{T-s}f(x_s)(XdB_s).$$

So:

$$\begin{aligned} & Ef(x_T)\int_0^T < dB_s, Y(TF_s(v_0)) > \\ = \; & E\int_0^T dP_{T-s}f\,(TF_s(v_0))\,ds \\ = \; & E\int_0^T (\delta P_{T-s})df(TF_s(v_0))ds \\ = \; & \int_0^T (\delta P_T)df(v_0)ds = T\; d(P_Tf)(v_0). \end{aligned}$$

End of the proof. ■

Note $\int_0^t E < dB_s, Y(T_xF_s(v)) >$ is a martingale if

$$\int_0^t E|Y(T_xF_s(v))|^2ds < \infty.$$

In terms of the metric on M determined by Y, this condition becomes:

$$\int_0^t E|T_xF_s|^2ds < \infty.$$

Corollary 8.2.3 *Let $p_t^h(x,y)$ be the heat kernel as defined on page 28, then*

$$\nabla \log p_t^h(\cdot, y_0)(x_0) = E\{\frac{1}{t}\int_0^t (TF_s)^*\,(X(x_s)dB_s)\,|x_t = y_0\}$$

under the assumptions of proposition 8.2.1.

Proof: The proof is just as for compact case. See [26]. Let $f \in C_K^\infty$. Differentiate equation (1.10) on page 23 to obtain:

$$d(P_t^hf)(v_0) = \int_M < \nabla p_t^h(-,y), v_0 >_{x_0} f(y)e^h dy.$$

On the other hand, we may rewrite equation (8.6) as follows:

$$d(P_t^h f)(v_0) = \int_M p_t^h(x_0, y) f(y) E\left\{\frac{1}{t}\int_0^t < TF_s(v_0), X(x_s)dB_s > |x_t = y\right\} e^h dy$$

Comparing the last two equations, we get:

$$\nabla p_t^h(-, y)(x_0) = p_t^h(x_0, y) E\left\{\frac{1}{t}\int_0^t TF_s^*(XdB_s)|x_t = y\right\}.$$

Thus finished the proof. ■

Remark: Assume $\int_0^t E|Y(T_x F_s)|^2 ds < \infty$ for each $x \in M$. If formula (8.6) holds for $f \in C_K^\infty$, it holds for $f \in C_0^2(M)$, the space of C^2 functions vanishing at infinity.

Proof: First assume f positive. Take f_n in C_K^∞ converging to f. Then $d(P_t f_n) \to d(P_t f)$ by Schauder type estimate as in the appendix. The convergence of the R.H.S. of the formula is also clear. Next take $f = f^+ - f^-$.

8.3 For higher order forms and gradient Brownian systems

Let α be a p form, β a 1 form, the wedge of α and β is a $p+1$ form defined as follows (following the notations from [1]):

$$(\alpha \wedge \beta)(v^1, \ldots, v^{p+1}) = \sum_{i=1}^{p+1} (-1)^{p+1-i} \beta(v^i) \alpha(v^1, \ldots, \widehat{v^i}, \ldots, v^{p+1}).$$

The symbol ˆ here means that the item below it is omitted.

The exterior differentiation of α is given by:

$$d\alpha(v_1, \ldots, v_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} \nabla\alpha(v_j)(v_1, \ldots, \widehat{v_j}, \ldots, v_{p+1}). \tag{8.7}$$

Let ϕ be a q form. Let $v_0 = (v_0^1, \ldots, v_0^q)$, $v_t = (TF_t(v_0^1), \ldots, TF_t(v_0^q))$. Analogously to the case of 1-forms, we define:

$$(\delta P_t)\phi(v_0) = E\phi(v_t)\chi_{t<\xi}. \tag{8.8}$$

Then by the argument in proposition 6.4.3, δP_t is a L^2 semigroup if $\sup_x E\left(|T_xF_t|^{2q}\chi_{t<\xi}\right) < \infty$.

Proposition 8.3.1 *Let M be a complete Riemannian manifold. Let (X, A) be a complete gradient Brownian system on M with generator $\frac{1}{2}\Delta^h$ and satisfying:*

$$E\left(\sup_{s\le t}|TF_s|^q\right) < \infty.$$

Then

$$(\delta P_t)\psi = e^{\frac{1}{2}t\Delta^h}\psi \tag{8.9}$$

for bounded q-forms ψ.

The proof is as in 6.4.1. Note a similar result holds for a gradient Brownian system with a general drift. In fact we could have a parallel discussion of the properties of δP_t for q forms as in chapter 6. We should also point out that as for 1-forms, extra conditions on TF_t will give (8.9) for gradient systems, without assuming nonexplosion(see corollary 7.2.3 on page 108).

Given a q form ψ, we define a q-1 form as follows:

$$\begin{aligned}\int_0^t \psi \circ dx_s(v_0) = &\ \tfrac{1}{q}\int_0^t \psi\left(X(x_s)dB_s, TF_s(v_0^1), \ldots, TF_s(v_0^{q-1})\right) \\ &-\tfrac{1}{2}\int_0^t \delta^h\psi\left(TF_s(v_0^1), \ldots, TF_s(v_0^{q-1}\right)\, ds.\end{aligned} \tag{8.10}$$

Here $v_0^q \in T_{x_0}M$, $i = 1, 2, \ldots, q$, and $v_0 = (v_0^1, \ldots, v_0^q)$.

Proposition 8.3.2 *Let (X, A) be a complete gradient Brownian system on a complete Riemannian manifold M with generator $\frac{1}{2}\triangle^h$. Suppose for a q form $\psi \in D(\triangle^{h,q}) \cap L^\infty$,*

$$(\delta P_t)\psi = e^{\frac{1}{2}t\triangle^h}\psi, \tag{8.11}$$

and also for each x in M,

$$E\int_0^t |T_xF_s|^{2q} ds < \infty.$$

Then:

$$P_t^{h,q}\psi(x_0) = (-1)^{q+1}\frac{1}{t}E\int_0^t \psi \circ dx_s \wedge \int_0^t < X(x_s)dB_s, TF_s(\cdot) > . \tag{8.12}$$

Proof: Let

$$Q_t(\psi)(v_0) = -\frac{1}{2}\int_0^t (\delta^h P_s^{h,q-1}\psi)(v_0)ds. \tag{8.13}$$

Notice $P_t^{h,q}(\psi)$ is smooth on $[0,T] \times M$ by elliptic regularity, so

$$\begin{aligned}
\frac{\partial}{\partial t}Q_t(\psi) &= -\frac{1}{2}\delta^h(P_t^h\psi), \\
d(Q_t(\psi)) &= -\frac{1}{2}\int_0^t d\delta^h(P_s^{h,q}\psi)ds, \\
\delta^h Q_t(\psi) &= -\frac{1}{2}\int_0^t \delta^h\delta^h(P_s^{h,q}\psi)\,ds = 0.
\end{aligned}$$

Moreover,

$$d(Q_t(\psi)) = \frac{1}{2}\int_0^t \triangle^{h,q}(P_s^{h,q}\psi)ds = P_t^{h,q}\psi - \psi.$$

since $\triangle^{h,q}\psi = -d\delta^h\psi$. Therefore:

$$\triangle^{h,q-1}(Q_t(\psi)) = -P_t^{h,q-1}(\delta^h\psi) + \delta^h\psi.$$

Next we apply Itô formula (see page 14) to $(t,v)| \to Q_{T-t}(\psi)(v)$:

$$\begin{aligned} Q_{T-t}\psi(v_t) = \; & Q_T\psi(v_0) + \textstyle\int_0^t \nabla Q_{T-s}\psi(X(x_s)dB_s)(v_s) \\ & + \textstyle\int_0^t Q_{T-s}\psi((d\wedge)^{q-1}(\nabla X(\cdot)dB_s))v_s) \\ & + \tfrac{1}{2}\textstyle\int_0^t \triangle^h Q_{T-s}\psi(v_s)ds + \int_0^t \frac{\partial}{\partial s}(Q_{T-s})\psi(v_s)ds. \end{aligned}$$

From the calculations above we get:

$$\begin{aligned} Q_{T-t}\psi(v_t) &= Q_T\psi(v_0) + \int_0^t \nabla Q_{T-s}\psi(X(x_s)dB_s)(v_s) \\ &+ \int_0^t Q_{T-s}\psi((d\wedge)^{q-1}(\nabla X(\cdot)dB_s))v_s) \\ &+ \frac{1}{2}\int_0^t \delta^h\psi(v_s)ds. \end{aligned}$$

Let $t = T$, then $Q_{T-t}(\psi) = 0$. By definition and the equality above, we have:

$$\begin{aligned} \textstyle\int_0^T \psi \circ dx_s(v_s) = \; & Q_T\psi(v_0) + \tfrac{1}{q}\textstyle\int_0^T \psi(X(x_s)dB_s, v_s) \\ & + \textstyle\int_0^T \nabla Q_{T-s}\psi(X(x_s)dB_s)(v_s) \\ & + \textstyle\int_0^T Q_{T-s}\psi((d\wedge)^{q-1}(\nabla X(\cdot)dB_s))v_s). \end{aligned} \tag{8.14}$$

We will calculate the expectation of each term of $\int_0^t \psi \circ dx_s$ after wedging with $\int_0^T < X(x_s)dB_s, TF_s(\cdot) > ds$. It turns out that the first term and the last term vanishes. The latter is from equation 1.2 for a gradient system on page 12.

Take $v_0 = (v_0^1, \ldots, v_0^q)$, write $v_s^i = TF_s(v_0^i)$.

Denote by $w_s(\cdot)$ the linear map:

$$w_s(\cdot) = \overbrace{(TF_s(\cdot), \ldots, TF_s(\cdot))}^{q-1}.$$

Then

$$E \quad \int_0^T \psi(X(x_s)dB_s, w_s(\cdot)) \wedge \int_0^T < X(x_s)dB_s, TF_s(\cdot) > (v_0)$$

$$
\begin{aligned}
&= \sum_{i=1}^{q}(-1)^{q-i}E\int_0^T \psi(v_s^i, v_s^1, \ldots, \widehat{v_s^i}, \ldots, v_s^q)\, ds \\
&= \sum_{i=1}^{q}(-1)^{q-i}(-1)^{i-1}E\int_0^T \psi(v_s^1, \ldots, v_s^q)\, ds \\
&= q(-1)^{q-1}E\int_0^T \psi(v_s^1, \ldots, v_s^q) ds \\
&= q(-1)^{q+1}\int_0^T P_s^h\psi(v) ds.
\end{aligned}
$$

The last step uses the assumption: $\int_0^t E|T_x F_s|^{2q} ds < \infty$. Similar calculation shows:

$$
\begin{aligned}
& E\{\int_0^T \nabla Q_{T-s}\psi(X(x_s)dB_s)(w_s(\cdot)) \wedge \int_0^T < X(x_s)dB_s, TF_s(\cdot) >\}(v) \\
= & \sum_{i=1}^{q}(-1)^{q-i}E\int_0^T \nabla(Q_{T-s}\psi)(v_s^i)(v_s^1, \ldots \hat{v_s^i}, \ldots, v_s^q)\, ds \\
= & (-1)^{q+1}E\int_0^T (d(Q_{T-s}\psi))(v_s^1, \ldots, v_s^q)\, ds, \\
= & (-1)^{q+1}\int_0^T P_s^h\left(P_{T-s}^h(\psi) - \psi\right)(v)\, ds \\
= & (-1)^{q+1}\left[T(P_T^h\psi)(v) - \int_0^T P_s^h\psi(v) ds\right].
\end{aligned}
$$

Comparing these with 8.14, we have:

$$
P_T^{h,q}\psi = (-1)^{q+1}\frac{1}{T}E\int_0^T \psi \circ dx_s \wedge \int_0^T < X(x_s)dB_s, TF_s(\cdot) > .
$$

End of the proof. ■ Note: With an additional condition: $\sup_{x\in M} E|T_x F_s|^{2q} < \infty$, the formula in the above proposition holds for forms which is not necessarily bounded. See the remark on page 129.

Corollary 8.3.3 *Let $\psi = d\phi$ be a C^2 q form, then*

$$P_t^{h,q}(d\phi) = (-1)^{q+1}\tfrac{1}{t}E\{\phi(\overbrace{TF_t(\cdot),\ldots TF_t(\cdot)}^{q-1}) \wedge \textstyle\int_0^t < X(x_s)dB_s, TF_s(\cdot) >\}. \tag{8.15}$$

Proof: In this case,

$$\int_0^t \psi \circ dx_s = \frac{1}{q}\int_0^t d\phi(X(x_s)dB_s, \overbrace{TF_s(\cdot),\ldots,TF_s(\cdot)}^{q-1}) + \frac{1}{2}\int_0^t \Delta^h\phi(\overbrace{TF_s(\cdot),\ldots,TF_s(\cdot)}^{q-1})\,ds.$$

There is also the following equality:

$$E\textstyle\int_0^t d\phi(X(x_s)dB_s, TF_s(\cdot),\ldots,TF_s(\cdot)) \wedge \int_0^t < X(x_s)dB_s, TF_s(\cdot) >$$
$$= qE\textstyle\int_0^t \nabla\phi\,(X(x_s)dB_s)\,TF_s(\cdot),\ldots,TF_s(\cdot)) \wedge \int_0^t < X(x_s)dB_s, TF_s(\cdot) > .$$

But by Itô formula,

$$\phi(v_t) = \phi(v_0) + \textstyle\int_0^t \nabla\phi(XdB_s)(v_s) + \frac{1}{2}\int_0^t \Delta^h\phi(v_s)ds.$$

Here v_t is the q-1 vector induced by TF_t as is defined in the beginning of the section. So

$$\int_0^t d\phi \circ dx_s \wedge \int_0^t < XdB_s, TF_s(\cdot) >$$
$$= E\{\phi(\overbrace{TF_t(\cdot),\ldots TF_t(\cdot)}^{q-1}) \wedge \int_0^t < X(x_s)dB_s, TF_s(\cdot) >\}.$$

End of the proof. ■

This corollary can also be proved directly as in theorem 8.2.2. The factor $\frac{1}{q}$ in the formula may look odd, but it is due to that the tensors concerned is not symmetric.

Chapter 9

Appendix

Lemma 9.0.4 *[5] Let M be a complete Riemannian manifold. There exists an increasing sequence $\{h_n\} \subset C_K^\infty$ such that:*

(1). $0 \leq h_n \leq 1$

(2). $\lim_{n\to\infty} h_n(x) = 1$, *each* x.

(3). $|\nabla h_n| \leq \frac{1}{n}$, *for all* n.

Proof: This is standard result. Here's a proof from [5]. Let $M = R^1$. We may construct such a sequence $\{f_n\}$ as is well known. For a complete Riemannian manifold, there is a C^∞ smooth function f on M such that $|\nabla f| \leq 1$ and $\{|f| \leq k\}$ is compact for all numbers k. Let $h_n = f_n \circ f$. Then h_n is an increasing sequence and satisfies the requirements.

Schauder type estimates

Let M be a smooth differential manifold. Let L be an elliptic differential operator on M. In local coordinates, we may write:

$$L = \sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b_i(x)\frac{\partial}{\partial x^i} + c.$$

Here $(a_{i,j}(x))$ is symmetric positive definite $n \times n$ matrix for each x. The coefficients are assumed to be C^2.

Consider the following differential equation of parabolic type:

$$Lu = \frac{\partial u}{\partial t} \tag{9.1}$$

on a domain $\Omega \subset R^{n+1}$. A solution u to equation (9.1) is a function which is jointly continuous in (t, x), C^2 in x and C^1 in t for $t > 0$. A function which satisfies the above regularity will be said to be in $C^{2,1}$.

Let D be a set of R^{n+1}, define:

$$|u|_{2,1}^{D} = \max_{D}(|u| + |D_x u| + |D_x^2 u| + |D_t u|). \tag{9.2}$$

Theorem 9.0.5 *Let B be a bounded domain in R^n, $D = B \times (0, 1)$, and u a $C^{2,1}$ solution to the parabolic equation. Let D_1 be a subdomain of D with $d(\partial D_1, \partial D)$ denoting the distance from the boundary of D_1 to the boundary of D. Then*

$$|u|_{2,1}^{D_1} \leq k \max_{D} |u|.$$

Here k is a constant depending only on the bounds of $a_{i,j}$, $da_{i,j}$, db_i, dc in D and $d(\partial D_1, \partial D)$.

See [36] P_{64} for reference. By standard argument in analysis on uniform convergence, we conclude (see [4] and P_{89} of [36]):

Let u_n be an increasing sequence of solutions of the parabolic equation (9.1). Assume $\lim_{n\to\infty} u_n = u$ pointwise in D. Then u is also a solution and the convergence is in fact uniformly in C^1 on compact subset D_1 of D. Thus $D_x u_n \to D_x u$, and $D_t u_n \to D_t u$ in D_1.

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Index